Contents

Chapter 1

What is a group?

1.1 Examples

This section contains a list of *algebraic structures* with different properties. Although these objects look different from each other, they do have some features in common, for example they are all equipped with algebraic operations (like addition, multiplication etc.). Several involve matrix multiplication. The properties of these operations can be studied and compared. An important theme of group theory (and all areas of abstract algebra) is the distinction between *structural* and *superficial* similarities and differences in algebraic structures - the meaning of this distinction will become precise as we continue, but hopefully these examples and comments can already give a sense of it.

1. $(\mathbb{Z}, +)$
   $\mathbb{Z}$ is the set of integers, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
   The "$+$" indicates that we are thinking of $\mathbb{Z}$ as being equipped with addition. This means that given any pair of integers a and b we can produce a new integer by taking their sum $a + b$. Moreover, the integer 0 has a *neutral* property with respect to addition - adding it to another integer has no effect on that other integer. Not only that, but every integer has a *negative* that is also an integer, such as 3 and -3, -105 and 105 etc. The relationship between an integer and its negative is that when we add them together we get the neutral element 0 - this means that the effect of adding 5 to some integer x can be undone by adding -5. In the language of group theory, the integers -5 and 5 are *inverses* of each other with respect to the addition operation on $\mathbb{Z}$.

2. $(\mathbb{C}^\times, \times)$
   Here $\mathbb{C}^\times$ denotes the set of *non-zero* complex numbers, and "$\times$" denotes multiplication of complex numbers. So for example

   $$(2 + 3i) \times (1 - i) = 5 + i.$$

   The product of two elements of $\mathbb{C}^\times$ is always an element of $\mathbb{C}^\times$ (we say that $C^\times$ is *closed* under multiplication of complex numbers). So "$\times$" is a *binary operation* on $\mathbb{C}^\times$.
   Note that the number 1 is neutral for multiplication on $\mathbb{C}^\times$ - multiplying by 1 has no effect on any complex number. Moreover, every element of $\mathbb{C}^\times$ has an *inverse* for multiplication, this is its reciprocal. For example the inverse of $1 + 2i$ is

   $$\frac{1}{1+2i} = \frac{1}{1+2i} \times \frac{1-2i}{1-2i} = \frac{1-2i}{5} = \frac{1}{5} - \frac{2}{5}i.$$

   The property that a complex number and its inverse have together is that their product is the neutral element 1 - this means that the effect of multiplying by one of them can be reversed by multiplying by the other. If we included the number 0 in our set we would lose this last property, since 0 does not have an inverse for multiplication in $\mathbb{C}$.

3. $(\mathrm{GL}(2,\mathbb{Q}), \times)$
Read this as "the general linear group of 2 by 2 matrices over the rational numbers" ("GL" stands for "general linear").

$$\mathrm{GL}(2,\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Q}; ad - bc \neq 0 \right\},$$

so we are talking about the set of 2 by 2 matrices that have rational entries and have non-zero determinant or equivalently that have inverses. The "$\times$" here stands for matrix multiplication. Note that if A and B are elements of $\mathrm{GL}(2,\mathbb{Q})$, then so also are their matrix products AB and BA (which might be not be the same).
Question: Is this obvious? Why is it true?
Question: What is the "neutral element" in this example? Does every element have an inverse?
Question: Why is attention restricted to the matrices with non-zero determinant in this example?

4. $(\{1, i, -i, -1\}, \times)$
Here we are talking about the set of complex fourth roots of unity, under multiplication of complex numbers. Note that this set is closed under multiplication, meaning that the product of any two elements of the set is again in the set. You can check this directly by writing out the whole multiplication table (a worthwhile exercise at this point). You can also idenify the neutral element and the inverse of each element. Note that this example (which involves a finite set) is a subset of the infinite example 2. above, with the same operation.

5. Let S_3 denote the following set of 3×3 matrices.

$$S_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

What happens when you mutiply two elements of S_3? Do you get an element of S_3 (in algebra language, is this set S_3 *closed under matrix multiplication*)? If so, is this an accident, or does it follow from some special property of the matrices in S_3? Does S_3 have a neutral element for multiplication? Does every element of S_3 have an inverse in S_3 for multiplication?

Now let S_4 denote the set of all permutations of the set $\{a, b, c, d\}$.
Recall that a *permutation* of the set $\{a, b, c, d\}$ is a bijective function from the set to itself. The permutation

$$\begin{array}{ccc} a & \longrightarrow & d \\ b & \longrightarrow & b \\ c & \longrightarrow & a \\ d & \longrightarrow & c \end{array}$$

is sometimes written as $\begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix}$.
Given two permutations σ and τ of $\{a, b, c, d\}$, we can *compose* them to form the functions $\sigma \circ \tau$ (σ after τ) and $\tau \circ \sigma$ (τ after σ). This composition works as for any functions and is often referred to as *multiplication* of permutations.

Claim: The functions $\sigma \circ \tau$ and $\tau \circ \sigma$ are again *permutations* of $\{a, b, c, d\}$.
Why is this true? What you have to do to answer this is show that these compositions are again *bijective* functions from $\{a, b, c, d\}$ to itself - this means that they take each of the four elements to a different image.

Question: Would you expect $\sigma \circ \tau$ and $\tau \circ \sigma$ to be the same function? If in doubt, try some examples.
Question: What is the connection between S_3 in the first part of this example and S_4 in the

second part? Are they closely related in some way that justifies giving them almost the same name? This is a question that is close to the heart of at least one important theme of group theory. Even if you don't have the language yet to articulate an answer to it, think about what kind of functions the matrices in S_3 might represent. Remember that matrices can be interpreted as linear transformations, and that matrix multipication then corresponds to composition of linear transformations.

Remark: To study group theory and abstract algebra, you may need to relax and expand your understanding of the meaning of the word *multiplication*. Multiplication of integers means something very specific: 5×7 is the number that you get from the addition $5 + 5 + 5 + 5 + 5 + 5 + 5$ or $7 + 7 + 7 + 7 + 7$ (why are these the same?). Mutiplication of real numbers (or complex numbers) are natural extensions of that. In advanced algebra the word "multplication" is often used for operations that don't resemble these familiar ones at all (this already happens in the case of matrix multiplication). It is a good idea to get used to thinking of the word multiplication as just meaning "a way of combining pairs of elements".

6. Let D_8 be the following set of 2×2 matrices.

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

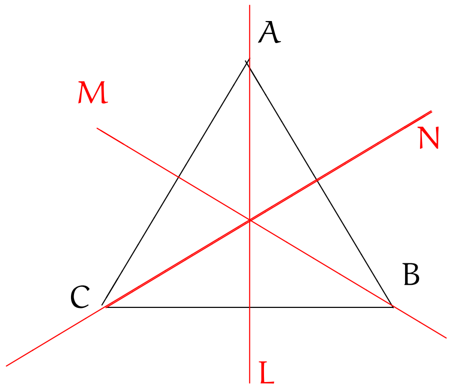
Note that D_8 is *closed under matrix multiplication*, the product of any pair of elements of D_8 is again in D_8. Also D_8 contains the inverse of each of its elements - check this. So D_8 is a "self-contained" algebraic structure - by taking its eight elements, multiplying them together as much as you like, and inverting them, you don't move outside the set D_8.

This example has a geometric interpretation. Think of each matrix in D_8 as a linear transformation of the plane $\mathbb{R}^2$, interpreted with respect to the standard basis $\{(1,0),(0,1)\}$. What do these particular transformations do? What do they do in particular to the square whose vertices are at the points $(1,0),(0,1),(-1,0)$ and $(0,-1)$? This is related to the next two examples, and to the last one.

7. **General groups of symmetries**
Suppose that P is some connected object in the two-dimensional plane, like a polygon or a line segment or a curve or a disc (*connected* means all in one piece). The following is an informal (and temporary) description of what is meant by a *symmetry* of P. Imagine that P is an object made of a rigid material. If you can pick up this piece of material from the plane and move it around (in 3-dimensional space) without breaking, compressing, stretching or deforming it in any way, and put it back so that the object occupies the same space that it originally did, you have implemented a symmetry of P.

For example, if P is a circular disc, then symmetries of P include rotations about the centre through any angle, reflections in any diameter, and any composition of operations of these kinds. Two symmetries are considered to be the same if P ends up in exactly the same position after both of them - for example in the case of the circular disc, a counter-clockwise rotation about the centre through a full $360°$ is the same as the rotation through $0°$ or the rotation through $720°$.

8. **Symmetries of an equilateral triangle**
Consider an equilateral triangle with vertices labelled A, B, C as in the diagram. For this example it does not matter whether you think of the triangle as consisting just of the vertices and edges or as a solid triangular disc.



The triangle has six symmetries:

- the identity symmetry I, which leaves everything where it is
- the counterclockwise rotation R_{120} through $120°$ about the centroid
- the counterclockwise rotation R_{240} through $240°$ about the centroid
- the reflections in the three medians: call these T_L, T_M, T_N.

Let D_6 denote the set of these six symmetries.
Note that the first three (the rotations) preserve the order in which the vertices A, B, C are encountered as you travel around the perimeter in a counter-clockwise direction; the last three (the reflections) change this order. If you think of the object as a "filled-in" disc, the reflections involve flipping it over and the rotations don't. (Note that the identity permutation is considered to be a rotation, through $0°$ - or any integer multiple of $360°$. It is certainly not a reflection).

Now that we have these six symmetries, we can compose pairs of them together.

Example: We define $R_{120} \circ T_L$ (read the "$\circ$" as "after") to be the symmetry that first reflects the triangle in the vertical line L and then applies the counter-clockwise rotation through $120°$. The overall effect of this leaves vertex B fixed and interchanges the other two, so it is the same as T_M - convince yourself of this, using a physical triangle if necessary. For every pair of our six symmetries, we can figure out what their composition is and write out the whole composition table, which is partly completed below. The entry in this table in the position whose row is labelled with the symmetry τ and whose column is labelled with the symmetry σ is $\tau \circ \sigma$.

$(D_6, \circ)$	I	R_{120}	R_{240}	T_L	T_M	T_N
I	I	R_{120}	R_{240}	T_L	T_M	T_N
R_{120}	R_{120}	R_{240}	I	T_M	T_N	T_L
R_{240}						
T_L	T_L	T_N	T_M	I	R_{240}	R_{120}
T_M						
T_N						

Important Exercise: By thinking about the compositions of all these symmetries, verify the part of the above table that is filled in and fill in the rest of it. You should find that each element of D_6 *appears exactly once in each row and in each column.*

One way to think about symmetries of the triangle is as geometric operations as above. Another is as permutations of the vertices. For example the reflection in the line L fixes the vertex A and swaps the other two, it corresponds to the permutation

$$\begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}.$$

The rotation R_{120} moves vertex A to the position of C, B to the position of A, and C to the position of B. It corresponds to the permutation

$$\left(\begin{array}{ccc} A & B & C \\ C & A & B \end{array}\right).$$

Another Important Exercise: Write down the permutations corresponding to the remaining elements of D_6 and verify that with this interpretation the composition of symmetries as defined above and the multiplication of permutations really amount to the same thing in this context (this means confirming that the permutation corresponding to the composition of two symmetries of the triangle is what you would expect based on the product of the two corresponding permutations).

Does *every* permutation of the vertices of the triangle arise from a symmetry? If so, what the second important exercise is really saying is that the set of symmetries of an equilateral triangle (with composition) is essentially the same object as the set of permutations of the set $\{A, B, C\}$, with permutation multiplication.

Part of our work in this course will be to precisely formulate what is meant by "essentially the same" here and to develop the conceptual tools and language to discuss situations like this. The examples in this section will hopefully be useful as our account of the subject becomes more technical and abstract.

So we have two interpretations of the set (or *group*) of symmetries of the triangle - as a collection of "moves"on the triangle and as a collection of permutations. Example 6 above shows that it could also be represented as a set of 2×2 matrices, linear transformations of $\mathbb{R}^2$.

9. Symmetries of a square
   Consider a square with vertices labelled A, B, C, D (in cyclic order as you travel around the perimeter). Let D_8 denote the set of symmetries of the square.

   Exercise: How many elements does D_8 have? Describe them in terms of rotations and reflections. Write down the permutation of $\{A, B, C, D\}$ corresponding to each one. Does every permutation of this set arise from a symmetry of the square? Can you figure out the details of the connection between this interpretation of D_8 and the one in Example 6 above?