Example 1

$\boxed{(\mathbb{Z},+)}$ addition of integers

$\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, 3, \ldots \}$

- We can add any pair of integers, the result is always an integer
- 0 has a special property for addition in $\mathbb{Z}$. Adding 0 to any integer has no effect. 0 is a neutral element or identity element for addition in $\mathbb{Z}$.
- Given any integer x, there is an additive inverse for x in $\mathbb{Z}$, namely $-x$.

e.g. -4 and 4 are additive inverses of each other, i.e. when we add -4 and 4 is the neutral element 0.

Example 2

$(\mathbb{C}^\times, \times)$
Here $\boxed{\mathbb{C}^\times}$ denotes the set of *non-zero* complex numbers, and "$\times$" denotes multiplication of complex numbers.

- If we multiply two elements of $\mathbb{C}^\times$, we get an element of $\mathbb{C}^\times$
- Neutral element : $1 \quad (= 1 + 0i)$
- Given any element z of $\mathbb{C}^\times$, does it have an inverse for multiplication in $\mathbb{C}^\times$?
  Given $z \in \mathbb{C}^\times$, is there a $z' \in \mathbb{C}^\times$ with
  $zz' = 1 \quad (= 1 + 0i)$?

Answer : Yes. If $z = a + bi$ then

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

$\boxed{\text{Why leave out } 0?}$

Example 3

2×2 entries are rational numbers

$(\mathrm{GL}(2,\mathbb{Q}), \times)$

Read this as "the general linear group of 2 by 2 matrices over the rational numbers" ("GL" stands for "general linear"). This time, $\times$ denotes matrix multiplication.

$GL(2,\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Q}, \quad ad-bc \neq 0 \right\}$

- $GL(2,\mathbb{Q})$ is closed under matrix multiplication
  If A, B are 2×2 matrices with $\det(A) \neq 0$, $\det(B) \neq 0$, then $\det(AB) \neq 0$ [since $\det(AB) = \det(A)\det(B)$]
- Identity element : $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad I_2 A = A I_2 = A$ always
- Inverses:
  If $A \in GL(2,\mathbb{Q})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
  $\boxed{AA^{-1} = I_2 = A^{-1}A}$ $A^{-1} \in GL(2,\mathbb{Q})$

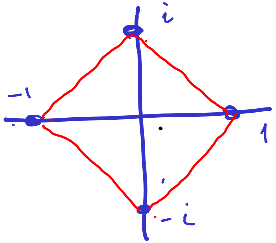
Example 4

complex multiplication

$$(\{1, i, -i, -1\}, \times)$$

Multiplication table

	1	i	$-i$	-1
1	1	i	$-i$	-1
i	i	-1	1	$-i$
$-i$	$-i$	1	-1	i
-1	-1	$-i$	i	1



(1) All already in our set!
$\{1, i, -i, -1\}$ is closed under multiplication

(2) 1 is a neutral element or identity element
(Multiplying by 1 has no effect)

(3) i and $-i$ are inverses of each other
$(i) \times (-i) = 1 = (-i) \times (i)$
1 is its own inverse
-1 is its own inverse

Example 5

Let S_3 denote the following set of 3×3 matrices.

$$S_3 = \left\{ \begin{array}{ccc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \right\}.$$

What happens when you mutiply two elements of S_3? Do you get an element of S_3 (in algebra language, is this set S_3 *closed under matrix multiplication*)? If so, is this an accident, or does it follow from some special property of the matrices in S_3? Does S_3 have a neutral element for multiplication? Does every element of S_3 have an inverse in S_3 for multiplication?

Challenge 2

Write

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc \neq 0 \right\}.$$

So S is the set of 2×2 matrices with integer entries and non-zero determinant.

Does every element of S have an inverse in S?
How can we quickly decide whether the inverse of a particular matrix in S is itself in S?
For example, what about $\begin{pmatrix} 3 & 2 \\ 8 & 5 \end{pmatrix}$?