MA3343 GROUPS 19/20 OUTLINE SOLUTIONS TO SAMPLE EXAM QUESTIONS

1. (a) Decide whether each of the following objects is a group. If your answer is *yes*, it is enough to just say yes. If you are answering *no* you should give a reason.
   i. The set of odd integers under addition.
   No - the set of odd integers is not closed under addition, since the sum of two odd integers is not odd.
   ii. The set of rational numbers of the form 2^i, where i is an integer, under multiplication.
   This is a group.

   (b) Give an example of
   i. A non-abelian group of order 6;
   D_6, the group of symmetries of an equilateral triangle.
   ii. An infinite non-abelian group;
   $GL(2,\mathbb{R})$, the group of 2×2 matrices with real entries and non-zero determinant, under matrix multiplication.
   iii. A finite group that has no element of order 3.
   D_8, the group of symmetries of the square.

   (c) If G is a group and x is an element of G, explain what is meant by the *cyclic subgroup* of G generated by x.
   The cyclic subgroup of G generated by x is the subgroup that consists of the identity element, x, x^{-1} and all elements of G that can be obtained by composing x with itself repeatedly or by composing x^{-1} with itself repeatedly.

   (d) Show that a group of even order must contain at least two elements whose square is equal to the identity.
   An element has square equal to the identity if and only if it is its own inverse. Every group has at least one such element, namely the identity. If the identity element is the only such element in some finite group G, then the number of non-identity elements in G must be even, since they can be organised into pairs, each with its inverse. Then the order of a finite group with only one element equal to its inverse must be odd, and every group of even order has at least two elements equal to their own inverses.

2. (a) State what is meant by the *centre* $Z(G)$ of a group G, and show that $Z(G)$ is a subgroup of G.
   The centre of G is the subset consisting of all those elements that commute with all elements of G. That $Z(G)$ is a subgroup of G is Theorem 2.2.3 of the lecture notes.

   (b) Suppose that H is a subgroup of G and that x and y are elements of G. Show that the left cosets xH and yH are either equal to each other or disjoint from each other.
   This is Lemma 2.1.7 in the lecture notes.

   (c) State Lagrange's Theorem on the order of a subgroup of a finite group, and use it to prove that every group of order 11 is cyclic.
   Lagrange's Theorem states that the order of a subgroup of a finite group is always a divisor of the order of the whole group. Let G be a group of order 11 and let x be a non-identity element of G. Let H be the cyclic subgroup of G generated by x. Then the order of H is at least 2, since H contains x and H contains the identity element of G. Also, $|H|$ divides 11 by Lagrange's Theorem. It follows that $|H| = 11$ and $H = G$. Hence G is cyclic, generated by x.

   (d) In S_9, let
   $$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 3 & 6 & 1 & 7 & 9 & 4 & 2 & 8 \end{pmatrix}.$$
   Determine the number of elements in the conjugacy class of π in S_9, and hence the order of the centralizer of π in S_9.
   $\pi = (1\,5\,7\,4)(2\,3\,6\,9\,8)$. The number of elements in the conjugacy class of π is
   $$\binom{9}{4} \times 3! \times 4! = \frac{9!}{20}.$$
   The order of the centralizer of π in S_9 is 20.

3. (a) Explain what is means to say that a group G *acts on* a set S. Explain the meaning of the terms *orbit* and *stabilizer* in connection with this, and state the Orbit-Centralizer Theorem.

   To say that G acts on S means that every element g of G determines a permutation of S that sends every $x \in S$ to $g \cdot x$. The system of permutations satisfies the following two properties:

   - $\mathrm{id}_G \cdot x = x$ for all $x \in S$;
   - $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and all $x \in S$.

   The orbit of an element x of S is $\{g \cdot x : g \in G\}$.
   The stabilizer of x is $\{g \in G : g \cdot x = x\}$.
   The Orbit-Stabilizer Theorem says that if x is an element of S whose orbit under the action of G is finite, then the number of elements in this orbit is the index in G of the stabilizer of x.

   (b) Let G be a group of order 16 acting on a set S of 15 elements. Show that there must be an element x of S with the property that $g \cdot x = x$ for all $g \in G$.

   Under the action of G, S is a disjoint union of distinct orbits. The number of elements in each orbit is a divisor of 16, so the possibilities are 1,2,4,8 and 16. Since it is not possible to write 15 as a sum of numbers from the list 1,2,4,8,16 without using the 1, there must be at least one orbit consisting of a single element x.

   (c) Explain what is meant by a *group homomorphism*. Show that the function $f : \mathbb{Z} \to \mathbb{R}^\times$ defined by $f(x) = 2^x$ is a homomorphism from the group of integers under addition to the group of non-zero real numbers under multplication.

   Let G and H be groups with operations $\star_G$ and $\star_H$ respectively. A group homomorphism from G to H is a function $\phi : G \to H$ that satisfies

$$\phi(x \star_G y) = \phi(x) \star_H \phi(y),$$

   for all elements x and y of G.
   Let $x, y \in \mathbb{Z}$. Then $f(x+y) = 2^{x+y} = 2^x \times 2^y = f(x) \times f(y)$.

   (d) What is meant by a *normal* subgroup of a group G?

   Let H be a subgroup of a group G. Suppose that $x \in H$, and that there is an element g in G for which $gxg^{-1} \notin H$. Prove that the left and right cosets gH and Hg are different subsets of G.

   A subgroup N of G is *normal* in G if for every $x \in G$, the left and right cosets xN and Nx are the same set.

   Since $x \in H$, the element gx is in the left coset gH. If gx is also in the right coset Hg. Then $gx = hg$ for some $h \in H$ and it follows that $gxg^{-1} = h$. Thus gx belongs to Hg only if $gxg^{-1} \in H$. Since $gxg^{-1} \notin H$, gx belongs to gH but not to Hg. Hence $gH \neq Hg$.