Group Actions

Definition 3.2.1 Let G be a group and let S be a set. We say that G acts on S if every element g of G induces a permutation π_g of the set S, subject to the following conditions

1. The identity element of G acts as the identity permutation on S - it leaves every element of S fixed.
2. If g and h are two elements of G, then the permutation π_{gh} of S is $\pi_g \circ \pi_h$.

Notation: If G is a group acting on a set S, let $g \in G$ and $x \in S$. The notation $g \cdot x$ is often used to refer to element of S that results from applying (the permutation determined by) g to the element x. Then condition 2. above says

$$g \cdot (h \cdot x) = gh \cdot x,$$

for all $g, h \in G$ and all $x \in S$.

Orbits and Stabilizers

Definition 3.2.2 The orbit of x under the action of G, denoted $O_G(x)$ or $G \cdot x$, is defined as the subset of S consisting of all elements that can be reached from x by applying elements of G.

$$G \cdot x = \{g \cdot x : g \in G\}.$$

Note that if G is finite, then the number of elements in the orbit of x is at most equal to the order of G, but it might be less.

Definition 3.2.2 The stabilizer in G of the element x of S, denoted $\mathbf{Stab}_G(x)$, is the subset of G consisting of those elements that leave x fixed.

$$\mathbf{Stab}_G(x) = \{g \in G : g \cdot x = x\}.$$

The Orbit-Stabilizer Theorem

Lemma 3.2.4 Let G be a group acting on a set S and let $x \in S$. Then $\textbf{Stab}_G(x)$ is a subgroup of G.

Theorem 3.2.5 (The Orbit-Stabilizer Theorem) Let G be a finite group acting on a set S, and let $x \in S$. Then the number of elements in the orbit $G \cdot x$ is equal to $[G : \textbf{Stab}_G(x)]$.

Cayley's Theorem

Let G be a group of order n, with elements $g_1, \ldots, g_n$. Let $g \in G$. Then $gg_1, \ldots, gg_n$ is a listing of the elements of G in some order, and the function

$$\begin{array}{ccc} g_1 & \longrightarrow & gg_1 \\ g_2 & \longrightarrow & gg_2 \\ \vdots & & \vdots \\ g_n & \longrightarrow & gg_n \end{array}$$

is a *permutation* of the elements of G, induced by left multiplication by the element g.

So every element of G gives rise to a (different) permutation of G and thus corresponds to an element of S_n. This correspondence establishes:

Theorem (Cayley's Theorem) *Every group of order n is isomorphic to some subgroup of S_n.*