Centralizers and Conjugacy

Definition 2.2.2 Let g be an element of a group $(G, \star)$. The centralizer of g in G, denoted $C_G(g)$, is the set of all elements of G that commute with g.

$$C_G(g) = \{x \in G : g \star x = x \star g\}.$$

Definition 2.2.4 Let G be a group and let $g \in G$. A conjugate of g in G is an element of G of the form xgx^{-1} for some $x \in G$.

Conjugacy classes

Fix $g \in G$ and consider the number of distinct conjugates of g in G. The element x of G determines the conjugate xgx^{-1} of G, but different choices of x can give the same element xgx^{-1}.

- $xgx^{-1} = g$ if and only if $xg = gx$, if and only if $x \in C_G(g)$.
- Also note that $xgx^{-1} = ygy^{-1}$ if and only if $y^{-1}xg = gy^{-1}x$, if and only if $y^{-1}x \in C_G(g)$.
- The statement that $y^{-1}x \in C_G(g)$ means exactly that $x = yh$ for some $h \in C_G(g)$, i.e. that x belongs to the left coset of $C_G(g)$ determined by y, and hence that the left cosets of $C_G(g)$ determined by x and y are the same.

The conclusion is that the number of distinct conjugates of g in G is the number of distinct left cosets of $C_G(g)$ in G.