MA3343: GROUPS
SEMESTER 1 2018-19
PROBLEM SHEET 3
Due date (for all problems): Friday November 30 2018

Please answer these questions as clearly and fully as you can. Start thinking about them as early as possible. If you are collaborating this time, please inform Rachel of the membership of your group by *November 23* and see the advice on the website about joint submissions. The numbers appearing in parentheses beside each problem link the problem to the learning outcomes for the course. [1]

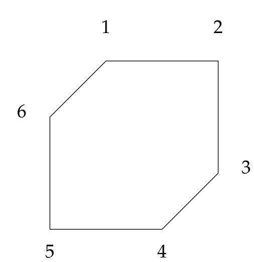
1. **(2)** In S_5, let
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}.$$
Calculate the products $\tau\sigma$ and $\sigma\tau$, and write each as a product of disjoint cycles.
(Please interpret $\tau\sigma$ as τ *after* σ, i.e. the permutation on the right in the product is applied first).

2. **(2,5,6)** How many conjugacy classes are in S_6?
For each conjugacy class, give the following information
   - An element of the class (express this as a product of disjoint cycles)
   - The number of elements in the class - give an explanation for this count.
   - The order of the centralizer of an element of the conjugacy class.

3. **(2,5,6)** How many elements of S_6 are equal to their own inverse?
Hint: If an element of S_6 has its square equal to the identity, what can its expression as a product of disjoint cycles look like?

4. **(2,5,6)** How many elements of S_5 have order 6? How many elements of S_6 have order 6?
Hint: recall that the order of an element in a symmetric group is the least common multiple of the orders (or lengths) of the cycles that appear in its expression as a product of disjoint cycles

5. **(2,5)** Let G be the group of symmetries of the figure shown, which consists of a square with identical isosceles triangles removed from opposite corners.



   (a) What are the elements of G? (Introduce notation to describe these elements as required.)
   (b) Write down the orbits of the vertex set $\{1, 2, 3, 4, 5, 6\}$ of the figure under the action of G.
   (c) Write down the stabilizer in G of each vertex.

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[1] LEARNING OUTCOMES
By the end of this course you will be able to :

1. Explain what a group is and use the definition of a group to identify examples and non-examples.
2. Use the language and terminology of group theory in an accurate and knowledgeable way.
3. Give examples of groups with certain specified properties.
4. State and prove some major theorems of group theory.
5. Identify and discuss important features of finite groups.
6. Critically assess proposed proofs of statements in group theory, and write some proofs of your own.

6. (2,5) If G is a finite group acting on a finite set S, then the number of orbits in S under the action is given by

$$\frac{1}{|G|}\sum_{g\in G} f(g),$$

where for $g \in G$, $f(g)$ is the number of elements x of S that satisfy $g \cdot x = x$.
Verify the above statement for the example of Question 5 above.

Note: the main thing that you have to do here is just read the statement carefully and make sure that you understand exactly what it is saying. If you can do this, verifying that it holds for the example is a straightforward task.

7. **(2,5,6)** Suppose that G is a group of order 49 acting on a finite set S whose number of elements is not a multiple of 7. Prove that there is some element x of S which is a *fixed point* for G, i.e. $g \cdot x = x$ for all $g \in G$.

Hint: Remember that S is partitioned into orbits by the action of G, and that the number of elements in an orbit is the index in G of a particular subgroup.

8. **(2,5,6)** Let $GL(2,\mathbb{Z})$ denote the group of all 2×2 matrices with integer entries and with determinant 1 or -1, and let $\mathbb{Z}^2$ denote the set of all column vectors of length 2 with integer entries. So $GL(2,\mathbb{Z})$ acts on $\mathbb{Z}^2$ by $A \cdot \binom{x}{y} = A\binom{x}{y}$, for a matrix A in $GL(2,\mathbb{Z})$.

   (a) What is the stabilizer in $GL(2,\mathbb{Z})$ of $\binom{1}{0}$?

   (b) Show that the orbit of $\binom{1}{0}$ under this action is the set of all vectors of the form $\binom{a}{b}$ where a and b are integers with $\gcd(a,b) = 1$.

   (c) What is the orbit of $\binom{2}{4}$?

   Hint: for (b). The orbit of $\binom{1}{0}$ is the set of all first columns of elements of $GL(2,\mathbb{Z})$.