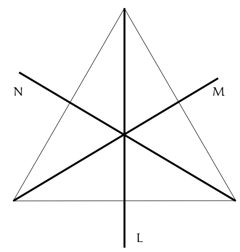
MA3343: GROUP THEORY
SEMESTER 1 2019-20
PROBLEM SHEET 2 - OUTLINE SOLUTIONS

1. Denote the six elements of D_6 by id, R_{120}, R_{240} (the three rotations) and T_L, T_M, T_N (the reflections in the lines L, M, N as shown).



   - $\mathrm{id}H = \{\mathrm{id}, R_{120}, R_{240}\} = H\mathrm{id}\ (= H)$.
   - $R_{120}H = \{\mathrm{id}, R_{120}, R_{240}\} = HR_{120}\ (= H)$
   - $R_{240}H = \{\mathrm{id}, R_{120}, R_{240}\} = HR_{240}\ (= H)$
   - $T_L H = \{T_L, T_M, T_N\} = HT_L$
   - $T_M H = \{T_L, T_M, T_N\} = HT_M$
   - $T_N H = \{T_L, T_M, T_N\} = HT_N$

   Since the left and right cosets coincide in all cases, the subgroup H is normal in D_6.

2. Write $K = \{\mathrm{id}, T_L\}$. Then

   - $R_{120}K = \{R_{120}, R_{120} \circ T_L\} = \{R_{120}, T_N\}$.
   - $KR_{120} = \{R_{120}, T_L \circ R_{120}\} = \{R_{120}, T_M\}$.

   Since the left and right cosets of K determined by the element R_{120} are different sets, K is not a normal subgroup of D_6.

3. That $Z(G)$ is a subgroup of $C_G(x)$ follows from the fact that every element of $Z(G)$ commutes with all elements of G, therefore in particular every element of $Z(G)$ commutes with x. To see that it is a *proper* subgroup, note that $x \in C_G(x)$ but $x \notin Z(G)$.
   That $C_G(x)$ is a proper subgroup of G follows from the fact that x is not in centre of G, so there are elements of G that don't commute with x and therefore don't belong to $C_G(x)$.
   Since $Z(G)$ is a proper subgroup of $C_G(x)$, the integer $\frac{|C_G(x)|}{|Z(G)|}$ is at least 2.
   Since $C_G(x)$ is a proper subgroup of G, the integer $\frac{|G|}{|C_G(x)|}$ is at least 2.
   Then the index of $Z(G)$ in G is not prime since it factorizes as

$$[G : Z(G)] = \frac{|C_G(x)|}{|Z(G)|} \times \frac{|G|}{|C_G(x)|}.$$

4. Prove that every group of order 7 is cyclic.

   Proof. Let G be a group of order 7 and let x be a non-identity element of G. Let H be the cyclic subgroup of G generated by x. Then $|H| > 1$ since $x \neq \mathrm{id}_G$. Then $|H| = 7$ by Lagrange's Theorem. Thus $H = G$ and G is cyclic. The same argument applies to any group of prime order. □

5. Using Problem 4 (if you wish), prove that every group of order less than 6 is abelian.

Proof. Every group of order 1 is abelian obviously. Because 2,3 and 5 are prime, groups of these orders are cyclic and hence abelian by Problem 4. Now suppose that G is a group of order 4, and let a, b, c be the non-identity elements of G. Each of these generates a cyclic subgroup of order 2 or 4. If any of them has order 4, then G is cyclic and hence abelian. The alternative is that the cyclic subgroups generated by a, b and c all have order 2, which means that $a^2 = b^2 = c^2 = \mathrm{id}$. In this case ab cannot be equal to id or to a or b and so must be equal to c. The same applies to ba. Similarly $ac = ca = b$ and $bc = cb = a$. Since all pairs of elements commute with each other G must be abelian. □

6. Let G be a group and let $x \in G$. Prove that $C_G(x)$, the centralizer of $x \in G$, is a subgroup of G.

   Proof. First observe that $\mathrm{id}_G x = x\mathrm{id}_G = x$ by definition. So id_G commutes with x and $\mathrm{id}_G \in C_G(x)$.
   Now suppose that $a, b \in C_G(x)$. Then $ax = xa$ and $bx = xb$. Hence

   $$(ab)x = abx = a(bx) = a(xb) = (ax)b = (xa)b = x(ab),$$

   and so $ab \in C_G(x)$ also. Thus $C_G(x)$ is closed under the group operation in G.
   Finally suppose $a \in C_G(x)$. Then $ax = xa$ and so

   $$a^{-1}axa^{-1} = a^{-1}xaa^{-1} \Longrightarrow xa^{-1} = a^{-1}x.$$

   Thus a^{-1} commutes with x and $a^{-1} \in C_G(x)$.
   Since $C_G(x)$ contains the identity element of G, is closed under the operation of G, and contains the inverse of each of its elements, it is a subgroup of G. □

7. (a) What is the centre of Q?
   Answer: $\{1, -1\}$
   (b) What is the centralizer in Q of the element -1?
   Answer: All of Q.
   (c) What is the centralizer in Q of the element i?
   Answer: $\{1, -1, i, -i\}$

8. As usual let $GL(2, \mathbb{R})$ denote the group of invertible 2×2 matrices with entries in $\mathbb{R}$, under the operation of matrix multiplication.

   (a) What is the centralizer in $GL(2, \mathbb{R})$ of the diagonal matrix with entries 2, 3 (in that order) along its main diagonal?
   Answer: The set of invertible diagonal matrices.
   (b) What is the centralizer in $GL(2, \mathbb{R})$ of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$?
   Answer: The set of matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where $a, b \in \mathbb{R}$ and $a \neq 0$.

9. An element of S_6 is equal to its own inverse if it is equal to the identity element or if the least common multiple of the cycles lengths in its expression as a product of disjoint cycles is 2. This means that the element is either a singe 2-cycle, a product of two disjoint 2-cycles, or a product of three disjoint 2-cycles.

   - Number of 2-cycles is $\binom{6}{2} = 15$.
   - Number of pairs of disjoint 2-cycles is $\binom{6}{4} \times 3 = 45$.
   - Number of products of three disjoint 2-cycles is $\frac{1}{3!} \times \binom{6}{2} \times \binom{4}{2} = 15$

   Number of elements equal to their own inverses is $1 + 15 + 45 + 15 = 91$.

10. In S_5, the elements of order 6 have cycle type consisting of one 2-cycle and one 3-cycle. The number of such elements is $2 \times \binom{5}{3} = 20$.
    In S_6, elements of order 6 either have a 2-cycle, a 3-cycle and a fixed point, or a single cycle of length 6. The number of the first type is $2 \times \binom{6}{3} \times \binom{3}{2} = 120$, and the number of the second type is 5!, also 120. The number of elements of order 6 is 240.