# MA3343/MA532: Groups
## Semester 1 2020-21
## Problem Sheet 2
Due date (for all problems): Friday December 4 2020

Please answer these questions marked $\star$ as clearly and fully as you can, and also think about the unmarked questions, which are relevant for independent study and exam preparation. The numbers appearing in parentheses beside each problem link the problem to the learning outcomes for the course. [1]

**Definition (left and right cosets, normal subgroup):** Let $H$ be a subgroup of a group $G$, and let $x \in G$. The *left coset* of $H$ determined by $x$ is the set

$$xH = \{xh : h \in H\}.$$

The right coset of $H$ determined by $x$ is the set

$$Hx = \{hx : h \in H\}.$$

It is possible for $xH$ and $Hx$ to be the same set, or not. If they are equal for *every* element $x$ of $G$, we say that $H$ is a *normal* subgroup of $G$.

**Note**: The concept of a normal subgroup is extremely important in group theory, although the reasons for that might not be clear yet.

1. **(2,5,6)** * Let $D_6$ be the group of symmetries of an equilateral triangle.
   Let $H$ be the subgroup of $D_6$ consisting of the three rotations.
   For each of the six elements of $D_6$, write down the left and right cosets of $H$ that it determines.
   Conclude that $H$ is a normal subgroup of $D_6$.

2. **(2,5,6)** * Let $K$ be the subgroup of $D_6$ consisting of the identity element and any one of the three reflections. By comparing the left and right cosets of $K$ determined by some suitably chosen element of $D_6$, show that $K$ is *not* a normal subgroup of $D_6$.
   (Note: to show that a subgroup is *not* normal, it is enough to show that there is just one element of the larger group whose left and right cosets are different. To know that a subgroup *is* normal, you need to know that the left and right cosets are the same for *all* elements of the group.)

3. **2,4,6** * Using the definition in Question 1 above, show that the centre of any group is a normal subgroup of the group. (You may assume that the centre is a subgroup and just explain why it is normal).

4. Let $G$ be a non-abelian finite group with centre $Z(G)$. Let $x$ be an element of $G$ that is not in $Z(G)$. Show that

   - $Z(G)$ is a *proper subgroup* of $C_G(x)$, and
   - $C_G(x)$ is a *proper subgroup* of $G$.

   Conclude that the index of $Z(G)$ in $G$ cannot be a prime number.

5. **(2,4,5,6)** Prove that every group of order 7 is cyclic.
   What is special about 7 here? What is the more general statement of which this is a special case?

6. **(2,5)** Using Problem 5 (if you wish), prove that every group of order less than 6 is abelian.

7. **(2,4,5,6)** Let $G$ be a group and let $x \in G$. Prove that $C_G(x)$, the centralizer of $x \in G$, is a subgroup of $G$.

8. **(1,5)** The *quaternion group* $Q$ of order 8 has elements $1, -1, i, -i, j, -j, k, -k$, with the following multiplication table.

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[1] Learning Outcomes
By the end of this course you will be able to :

1. Explain what a group is and use the definition of a group to identify examples and non-examples.
2. Use the language and terminology of group theory in an accurate and knowledgeable way.
3. Give examples of groups with certain specified properties.
4. State and prove some major theorems of group theory.
5. Identify and discuss important features of finite groups.
6. Critically assess proposed proofs of statements in group theory, and write some proofs of your own.

| | 1 | −1 | i | −i | j | −j | k | −k |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | −1 | i | −i | j | −j | k | −k |
| −1 | −1 | 1 | −i | i | −j | j | −k | k |
| i | i | −i | −1 | 1 | k | −k | −j | j |
| −i | −i | i | 1 | −1 | −k | k | j | −j |
| j | j | −j | −k | k | −1 | 1 | i | −i |
| −j | −j | j | k | −k | 1 | −1 | −i | i |
| k | k | −k | j | −j | −i | i | −1 | 1 |
| −k | −k | k | −j | j | i | −i | 1 | −1 |

(a) What is the centre of Q?

(b) What is the centralizer in Q of the element −1?

(c) What is the centralizer in Q of the element i?

9. **(2,5)** * As usual let $GL(2,\mathbb{R})$ denote the group of invertible $2 \times 2$ matrices with entries in $\mathbb{R}$, under the operation of matrix multiplication.

   (a) What is the centralizer in $GL(2,\mathbb{R})$ of the diagonal matrix with entries $2, 3$ (in that order) along its main diagonal?

   (b) What is the centralizer in $GL(2,\mathbb{R})$ of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$?

10. **(2,5,6)** * How many elements of $S_7$ are equal to their own inverse?

11. How many elements of $S_5$ have order 6? How many elements of $S_6$ have order 6?

12. **(2,5,6)** * How many elements of $S_{11}$ have order 15?

REMARKS ON THE PROBLEMS

1.,2. This is a matter of carefully reading the definitions at the top of the page and making sure you understandthem, than applying them to the case of $D_6$ with the subgroups H and K. Note that to show that a subgroup is *not* normal, it is enough to show that there is just one element of the larger group whose left and right cosets are different. To know that a subgroup *is* normal, you need to know that the left and right cosets are the sameforallelements of the group. So there is potentially more checking to do for Problem 1 than for Problem 2.

4. This has been discussed a bit in the video lectures from Week 5.

5. Suppose G is a group of order 7 and let x be a non-identity element of G. Think about the cyclic subgroup of G that is generated by x. What can its order be?

6. Groups of order 4 are the only problem here really, because of Problem 5. Let G be a group of order 4 and let $a, b, c$ be its non-identity elements. The cyclic subgroup generated by each of these elements must have order 2 or 4 (why?). Either all of them have order 2 or one of them is all of G. Write out the group table for each case

10. An element of $S_7$ is equal to its own inverse if it is the identity or if the least common multiple of the cycle lengths in its description as a product of disjoint cycles is 2.

11.,12. Think about the possible arrangements of disjoint cycles in the relevant symmetric groups that would have the required least common multiple, and then count the elements in each of the relevant classes.