MA3343/MA532: GROUPS
SEMESTER 1 2019-20
PROBLEM SHEET 2
Due date (for questions marked with $*$): Friday November 8 2016

Please answer as clearly and fully as you can, and also think about the unmarked questions, which are relevant for independent study and exam preparation. If you have collaborators this time, please inform Rachel of the membership of your group by *November 1* and see the advice on the Blackboard page about joint submissions. We can discuss these at the tutorials in the weeks of October 29 and November 4. The numbers appearing in parentheses beside each problem link the problem to the learning outcomes for the course. [1]

Definition (left and right cosets, normal subgroup): Let H be a subgroup of a group G, and let $x \in G$.
The *left coset* of H determined by x is the set

$$xH = \{xh : h \in H\}.$$

The right coset of H determined by x is the set

$$Hx = \{hx : h \in H\}.$$

It is possible for xH and Hx to be the same set, or not. If they are equal for *every* element x of G, we say that H is a *normal* subgroup of G.

Note: The concept of a normal subgroup is extremely important in group theory, although the reasons for that might not be clear yet.

1. **(2,5,6)** $*$ Let D_6 be the group of symmetries of an equilateral triangle.
   Let H be the subgroup of D_6 consisting of the three rotations.
   For each of the six elements of D_6, write down the left and right cosets of H that it determines.
   Conclude that H is a normal subgroup of D_6.
2. $*$ Let K be the subgroup of D_6 consisting of the identity element and any one of the three reflections. By comparing the left and right cosets of K determined by some suitably chosen element of D_6, show that K is *not* a normal subgroup of D_6.
3. $*$ Let G be a non-abelian finite group with centre $Z(G)$. Let x be an element of G that is not in $Z(G)$. Show that
   - $Z(G)$ is a *proper subgroup* of $C_G(x)$, and
   - $C_G(x)$ is a *proper subgroup* of G.

   Conclude that the index of $Z(G)$ in G cannot be a prime number.
4. **(2,4,5,6)** Prove that every group of order 7 is cyclic.
   What is special about 7 here? What is the more general statement of which this is a special case?
5. **(2,5)** Using Problem 4 (if you wish), prove that every group of order less than 6 is abelian.
6. **(2,4,5)** Let G be a group and let $x \in G$. Prove that $C_G(x)$, the centralizer of $x \in G$, is a subgroup of G.
7. **(5)** $*$ The *quaternion group* Q of order 8 has elements $1, -1, i, -i, j, -j, k, -k$, with the following multiplication table.

[1] LEARNING OUTCOMES
By the end of this course you will be able to :

1. Explain what a group is and use the definition of a group to identify examples and non-examples.
2. Use the language and terminology of group theory in an accurate and knowledgeable way.
3. Give examples of groups with certain specified properties.
4. State and prove some major theorems of group theory.
5. Identify and discuss important features of finite groups.
6. Critically assess proposed proofs of statements in group theory, and write some proofs of your own.

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

(a) What is the centre of Q?

(b) What is the centralizer in Q of the element -1?

(c) What is the centralizer in Q of the element i?

8. **(5)** $*$ As usual let $GL(2,\mathbb{R})$ denote the group of invertible 2×2 matrices with entries in $\mathbb{R}$, under the operation of matrix multiplication.

   (a) What is the centralizer in $GL(2,\mathbb{R})$ of the diagonal matrix with entries $2, 3$ (in that order) along its main diagonal?

   (b) What is the centralizer in $GL(2,\mathbb{R})$ of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$?

9. **(2,5,6)** How many elements of S_6 are equal to their own inverse?

10. **(2,5,6)** $*$ How many elements of S_5 have order 6? How many elements of S_6 have order 6?

REMARKS ON THE PROBLEMS

1.,2. This is really a matter of carefully reading the definitions at the top of the page and making sure you understand them, than applying them to the case of D_6 with the subgroups H and K. Note that to show that a subgroup is *not* normal, it is enough to show that there is just one element of the larger group whose left and right cosets are different. To know that a subgroup *is* normal, you need to know that the left and right cosets are the same for *all* elements of the group. So there is potetially more checking to do for Problem 1 than for Problem 2.

3. This might be challenging, especially the last part of it. Versions of it have appeared on exam papers in this module in the last few years. Remember that a *proper* subgroup is a subgroup that is not the whole group. This is a matter of thinking carefully about the concept of the centralizer of an element. Why must the centralizer of the element x contain the centre of G? Why can't it be equal to the centre of G (what else must be there)? Why can't the centralizer of x be all of G?
   For the last part, think about Lagrange's Theorem applied to the situation where A is a subgroup of B and B is a subgroup of C. Remember that in this case A is also a subgroup of C.

4. Suppose G is a group of order 7 and let x be a non-identity element of G. Think about the cyclic subgroup of G that is generated by x. What can its order be?

5. Groups of order 4 are the only problem here really, because of Problem 4. Let G be a group of order 4 and let a, b, c be its non-identity elements. The cyclic subgroup generated by each of these elements must have order 2 or 4 (why?). Either all of them have order 2 or one of them is all of G. Write out the group table for each case.

6. You can follow an argument very similar to the one in the proof of Theorem 2.2.3 in the lecture notes. Remember that you are interested in elements that commute with x, not necessarily with everything in G.

7. This group is related to Hamilton's famous quaternion division algebra. Every non-abelian group of order 8 has identical structure either to Q_8 or to the dihedral group D_8.

8. Write down a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and figure out under what conditions on a, b, c, d this commutes with the matrix of part (a) and the matrix of part (b).

9. If an element of S_6 has its square equal to the identity, what can its expression as a product of disjoint cycles look like?

10. Again the question here is what the disjoint cycle structure of an element of order 6 can look like. The answer to that is not the same for S_5 and S_6.