MA3343/MA532: GROUPS
SEMESTER 1 2018-19
PROBLEM SHEET 2
Due date (for all problems): Friday November 9 2018

Please answer these questions as clearly and fully as you can. Start thinking about them as early as possible. If you have collaborators this time, please inform Rachel of the membership of your group by *November 2* and see the advice on the website about joint submissions. The numbers appearing in parentheses beside each problem link the problem to the learning outcomes for the course. The main themes that are involved in this assignment are:

- Cyclic subgroups and generating sets (from the end of Chapter 1)
- Cosets and Lagrange's Theorem (from Section 2.1)
- Centralizers and conjugacy classes (from Section 2.2)

Some of the tasks deal with more than one of these themes of course. If you are not already clear on the definitions of all of these items, hopefully you will be after working on these problems. [1]

1. **(2,5,6)**

   (a) Let D_6 be the group of symmetries of an equilateral triangle. Show that D_6 can be generated by one rotation and one reflection, or by two reflections.

   (b) Is it true that *any* choice of two reflections in the group D_8 of symmetries of the square will generate the whole group D_8?

   Note: This should hopefully not be too difficult, what is mainly required is patience and careful writing. The first important step is to set up your notation in a clear way, so that you (and your reader) can see what you are doing. Then it is a matter (in part (a)) of choosing two elements of the required type and showing that from them you can generate all the remaining elements.

   Part (b) is different - think about the different pairs of reflections that you could choose. If you can find one pair of (distinct) reflections that *doesn't* generate D_8, that's all you need to do.

2. **(2,4,5,6)** Prove that every group of order 7 is cyclic.
   What is special about 7 here? What is the more general statement of which this is a special case?

   Hint: Suppose G is a group of order 7 and let x be a non-identity element of G. Think about the cyclic subgroup of G that is generated by x. What can its order be?

3. **(2,5)** Using Problem 2 (if you wish), prove that every group of order less than 6 is abelian.

   Groups of order 4 are the only problem here really, because of Problem 2. Let G be a group of order 4 and let a, b, c be its non-identity elements. The cyclic subgroup generated by each of these elements must have order 2 or 4 (why?). Either all of them have order 2 or one of them is all of G. Write out the group table for each case.

4. **(2,5)** Let H be a subgroup of a finite group G, and let $x, y \in G$. Prove that x and y belong to the same left coset of H in G if and only if $x^{-1}y \in H$.
   There are several ways to organise this. Note that it is an "if and only if" statement, which means it is really two statements, both of which need to be proved (they are converses of each other). Remember that all you need to do to show that two cosets are the same is show that they have non-empty intersection. It may be helpful to note also that x belongs to the coset xH and that y belongs to the coset yH.

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[1] LEARNING OUTCOMES
By the end of this course you will be able to :

1. Explain what a group is and use the definition of a group to identify examples and non-examples.
2. Use the language and terminology of group theory in an accurate and knowledgeable way.
3. Give examples of groups with certain specified properties.
4. State and prove some major theorems of group theory.
5. Identify and discuss important features of finite groups.
6. Critically assess proposed proofs of statements in group theory, and write some proofs of your own.

5. **(1,2,5)** Let $G = GL(2,\mathbb{Q})$, the group of nonsingular 2×2 matrices with rational entries, under matrix mutiplication.

   (a) Let H be the subset of G consisting of all matrices in G with $\binom{1}{0}$ in the first column. Show that H is a subgroup of G. How can you tell by glancing at a pair of elements of G whether they belong to the same left coset of H?

   (b) For $A \in G$, the *right coset* of H determined by A is the set of all matrices of the form BA where $B \in H$. How can you tell by glancing at a pair of elements of G whether they belong to the same right coset of H?

   (c) Give an example of

      i. A pair of elements of G that belong to the same left coset of H and to the same right coset of H.
      ii. A pair of elements of G that belong to the same left coset of H but to different right cosets of H.
      iii. A pair of elements of G that belong to different left cosets of H but to the same right coset of H.
      iv. A pair of elements of G that belong to different left cosets of H and to different right cosets of H.

   The point of this question is to think about the concept of coset in the context of a reasonably concrete example. The fact that the left and right coset determined by a particular element may differ is something that we will revisit in Chapter 4, it is related to the concept of a *normal subgroup*. For part (a), one way to approach it is to start with some specific "typical" matrix in G and look at the elements of the left coset of H that it determines. How to they resemble each other? Part (b) is similar. If you have figured out what is happening in (a) and (b), then (c) should be straightforward - the point there really is to observe that all four of these situtations can occur.

6. **(2,5)** Let G be a group and let $x \in G$.
   Definition: The *centralizer* of x in G, denoted $C_G(x)$, is defined to be the set of elements of G that commute with x, i.e.

$$C_G(x) = \{y \in G : yx = xy\}.$$

   (a) The *quaternion group* Q of order 8 has elements $1, -1, i, -i, j, -j, k, -k$, with the following multiplication table.

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

      i. What is the centre of Q?
      ii. What is the centralizer in Q of the element -1?
      iii. What is the centralizer in Q of the element i?

   (b) As usual let $GL(2,\mathbb{R})$ denote the group of invertible 2×2 matrices with entries in $\mathbb{R}$, under the operation of matrix multiplication.

      i. What is the centralizer in $GL(2,\mathbb{R})$ of the diagonal matrix with entries $2, 3$ (in that order) along its main diagonal?
      ii. What is the centralizer in $GL(2,\mathbb{R})$ of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$?

   Note: The first thing to do here is make sure you understand the definition of centralizer. Making sense of this definition and being able to use it to analyse the examples is the main point of this problem. For part (a), a suitable approach might be to inspect the whole multiplication table and identify those elements that commute with your particular element in each of i, ii, iii.

   For part (b), one suggestion is to write down a completely general 2×2 matrix and figure out under what conditions on its entries it will commute with the matrices in parts i and ii.

7. **(2,4,5,6)** Let G be a group and let $x \in G$. Prove that $C_G(x)$, the centralizer of $x \in G$, is a subgroup of G.

8. **(2,4,5,6)** Let G be a finite group. Prove that the index of $Z(G)$ in G cannot be a prime number.

   This one is maybe challenging. Here is one way to approach it - first deal with the case where G is abelian (what is the index of $Z(G)$ in G in this case?). If G is not abelian, think about the centralizer of an element of G that does not belong to the centre. Use Lagrange's Theorem.