Group Theory (MA3343): Semester I 2020-2021
Exam Information and Advice

January 11, 2021

1 Format and General Advice

1. The exam will consist of four questions, of which you must answer any three for full marks. Each question has four parts. The parts all carry equal marks. It will be a two-hour open book exam, with submission via Blackboard.

2. The syllabus for the course consists of the lecture notes, video lectures, and the problem sheets.

3. Questions 1 and 2 correspond respectively (more or less) to Chapter 1 and to Sections 2.1 and 2.2 of the lecture notes. Question 3 corresponds (more or less) to Section 2.3 and Chapter 3 of the lecture notes. Question 4 corresponds to Chapter 4 of the lecture notes.

4. Make sure that you are clear on the meanings of the technical terms that we encountered in the course, so that you don't risk losing out by misinterpreting a question or by giving an example of the wrong kind. For example, make sure you know the meaning of terms like abelian, order, coset, subgroup, cyclic, index, conjugate, conjugacy class, centre, centralizer, disjoint cycles, transpositions, odd and even permutations, homomorphism, normal subgroup and so on. (These are just examples but the list of technical group theory terms that we have used is not that long).

5. Make sure that you know what is required to show that some subset of a group is a subgroup.

6. This is an "open book" exam. But answers that are copied *verbatim* from online resources or notes will not receive high scores. As the examiner I am looking to see that students understand what they are writing, so it is best to write in your own words, as you would do in a typical exam. Students are not allowed to collaborate on exam questions.

   If you access a website, or notes taken at a tutorial, or our own lecture notes when answering any question (which is not discouraged at all!), you should say so in your answer to that question.

7. The style of questions will be generally similar to recent years, but there are some adaptations due to the open book format (so there won't be questions that can be answered by transcribing from the lecture notes). With this in mind, I have included some sample "exam-type" questions below. We can discuss these in workshops in advance of the exam.

8. At the exam, make sure you read the questions very carefully and that you are completely clear on what is asked before you start writing your answer. This may seem like a facile piece of advice, but it happens frequently that students misinterpret or misread questions while under exam pressure, and waste a lot of time.

2 Sample Questions

These sample questions are intended to give you an idea of the style and length of question to expect in the exam. In terms of content, please do not overanalyze the sample questions or read too much into them.

MA3343 GROUPS

Sample Paper (for two-hour online open-book exam)

1. (a) Answer TRUE or FALSE to each of the following (there is no need to provide explanation).
   - i. The binary operation in a group must be commutative.
   - ii. A group could have more than one identity element.
   - iii. The binary operation in a group must be associative.
   - iv. If n is a positive integer, then there exists a group that has exactly n elements.
   - v. The set of integers contains an identity element for the operation of subtraction.
   - vi. There exists a non-abelian group with exactly 17 elements.

   (b) Determine whether each of the following objects is a group, with the indicated operation. In each case, if your answer is that the object *is* a group, it is enough to just say so. If not, please give at least one reason why not.
   - i. The set of all 2×2 matrices with integer entries, under addition.
   - ii. The set of all 2×2 matrices with integer entries and non-zero determinant, under matrix multiplication.
   - iii. The set of all 2×2 matrices with real entries, under matrix multiplication.

   (c) Explain what it means for a group to be *abelian*. Give an example of
   - i. An abelian group with eight elements;
   - ii. A finite non-abelian group;
   - iii. An infinite abelian group;
   - iv. An infinite non-abelian group.

   (d) Suppose that S is a subset of a group G. What does it mean to say that S *generates* G? In the general linear group $GL(2, \mathbb{R})$, let

   $$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Write down all the elements of the subgroup of $GL(2, \mathbb{R})$ that is generated by A and B.

2. (a) Let H be a subgroup of a group G, and let x and y be elements of G. Show that x and y belong to the same left coset of H in G if and only if $x^{-1}y \in H$.

   (b) Let H denote the subgroup of $GL(2, \mathbb{R})$ consisting of all diagonal matrices with non-zero entries on the main diagonal. Exactly two of the following matrices belong to the same left coset of H in $GL(2, \mathbb{R})$. Which ones?

   $$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \; \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}, \; \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}, \; \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

   (c) Let G be a finite group and let $Z(G)$ denote the centre of G. Stating any theorem that you need to use, explain why the index of $Z(G)$ in G cannot be prime.

   (d) Suppose that x and y are elements of a group G with $xy \neq yx$. Show that $C_G(x)$ and $C_G(y)$ are different subgroups of G.

3. (a) Let π denote the element

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 6 & 2 & 3 \end{pmatrix}$$

of the symmetric group S_6. How many distinct conjugates does π have in S_6?

(b) Give an example of

i. An action of a finite group on a finite set, with a single orbit.

ii. An action of a finite group on a set of six elements, with two orbits of two elements each.

iii. An action of a finite group on a finite set, with exactly two orbits, of different sizes.

(c) A group of order 21 acts on a set with 17 elements. There are exactly three orbits. Determine the numbers of elements in the three orbits and explain your reasoning, with reference to any theorem that you need to use.

(d) What does Cayley's Theorem say about the group of rotational symmetries of the regular hexagon? Determine whether this group is isomorphic to a subgroup of the symmetric group S_4.

4. (a) Let G and H be groups. What does it mean to say that a function $\phi : G \to H$ is a group homomorphism?
Determine, with explanation, whether each of the following functions is a homomorphism from the group $\mathbb{R}^+$ of positive real numbers under multiplication, to itself.

i. $f_1 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f_1(x) = e^x$, for $x \in \mathbb{R}^+$.

ii. $f_2 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f_2(x) = x^3$, for $x \in \mathbb{R}^+$.

iii. $f_3 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f_3(x) = 2x$, for $x \in \mathbb{R}^+$.

(b) Suppose that H is a subgroup of a group G, with the property that for every element h of H and every element g of G, the element ghg^{-1} belongs to H. Let x be an element of G. Prove that the left and right cosets xH and Hx are equal.

(c) Give an example of a proper nontrivial normal subgroup of $GL(3, \mathbb{R})$ and an example of a proper nontrivial subgroup of $GL(3, \mathbb{R})$ that is not normal.

(d) Let G be a group and suppose $x \in G$. Is it true that $C_G(x)$ must be a normal subgroup of G? Explain your answer.