MA343: GROUP THEORY
SEMESTER 1 2018-19
PROBLEM SHEET 3 - OUTLINE SOLUTIONS AND COMMENTS
Please note that these are not intended to be interpreted as complete solutions.

1. **(2)** In S_5, let
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}.$$
Calculate the products $\tau\sigma$ and $\sigma\tau$.
(Please interpret $\tau\sigma$ as τ *after* σ, i.e. the permutation on the right in the product is applied first).
$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix} = (1\,3)(2\,4).$$
$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix} = (1\,5)(3\,4).$$

2. **(2,5,6)** How many conjugacy classes are in S_6?
For each conjugacy class, give the following information
   - An element of the class (express this as a product of disjoint cycles)
   - The number of elements in the class - give an explanation for this count.
   - The order of the centralizer of an element of the conjugacy class.

   There are 11 conjugacy classes in S_6. The following table gives a summary (not a complete solution of course).

Partition	Sample element	No. of elements	Order of centralizer
$1+1+1+1+1+1$	id	1	720
$1+1+1+1+2$	$(1\,2)$	15	48
$1+1+1+3$	$(1\,2\,3)$	40	18
$1+1+2+2$	$(1\,2)(3\,4)$	45	16
$1+2+3$	$(1\,2\,3)(4\,5)$	120	6
$1+1+4$	$(1\,2\,3\,4)$	90	8
$1+5$	$(1\,2\,3\,4\,5)$	144	5
$2+2+2$	$(1\,2)(3\,4)(5\,6)$	15	48
$3+3$	$(1\,2\,3)(4\,5\,6)$	40	18
$2+4$	$(1\,2)(3\,4\,5\,6)$	90	8
6	$(1\,2\,3\,4\,5\,6)$	120	6

3. **(2,5,6)** How many elements of S_6 are equal to their own inverse?
An element of S_6 is equal to its own inverse if its expression as a product of disjoint cycles consists only of fixed points and transpositions (i.e. cycles of lengths 1 and 2). There are four classes with this property in S_6, together they account for $1 + 15 + 45 + 15 = 76$ elements.
Note: some people answered with the number of relevant conjugacy classes (which is 4) rather than the number of elements.

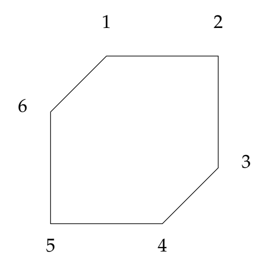
LEARNING OUTCOMES
By the end of this course you will be able to :

1. Explain what a group is and use the definition of a group to identify examples and non-examples.
2. Use the language and terminology of group theory in an accurate and knowledgeable way.
3. Give examples of groups with certain specified properties.
4. State and prove some major theorems of group theory.
5. Identify and discuss important features of finite groups.
6. Critically assess proposed proofs of statements in group theory, and write some proofs of your own.

4. **(2,5,6)** How many elements of S_5 have order 6?
Elements of S_5 have order 6 exactly if they belong to the class corresponding to the partition $2+3$. There are 20 such elements.

How many elements of S_6 have order 6?
Elements of S_6 have order 6 if they belong to the class corresponding to the partition $1+2+3$ (i.e. one 3-cycle and one transposition) or the class corresponding to the partition 6 (i.e. a single 6-cycle). The total number of such elements is $120 + 120 = 240$.

5. Let G be the group of symmetries of the figure shown, which consists of a square with identical isosceles triangles removed from opposite corners.



(a) What are the elements of G? (Introduce notation to describe these elements as required.)
id, rotation through 180°, reflection T_L in the axis that passes through the vertices 2 and 5, and reflection T_M in the perpendicular bisector of the line segment joining the vertices 2 and 5.

(b) Write down the orbits of the vertex set $\{1, 2, 3, 4, 5, 6\}$ of the figure under the action of G.
$\{1, 3, 4, 6\}$ and $\{2, 5\}$.

(c) Write down the stabilizer in G of each vertex.
$\mathrm{Stab}_G(1) = \mathrm{Stab}_G(3) = \mathrm{Stab}_G(4) = \mathrm{Stab}_G(6) = \{\mathrm{id}\}$.
$\mathrm{Stab}_G(2) = \mathrm{Stab}_G(5) = \{\mathrm{id}, T_L\}$.

6. **(2,5,6)** If G is a finite group acting on a finite set S, then the number of orbits in S under the action is given by

$$\frac{1}{|G|}\sum_{g\in G} f(g),$$

where for $g \in G$, $f(g)$ is the number of elements x of S that satisfy $g \cdot x = x$.
Verify the above statement for the example of Question 5 above.

In this example we have $f(\mathrm{id}) = 6$, $f(T_L) = 2$ and $f(g) = 0$ for all other elements g of the group. Thus

$$\frac{1}{|G|}\sum_{g\in G} f(g) = \frac{1}{4}(2+6) = 2,$$

which is the number of orbits.

7. Suppose that G is a group of order 49 acting on a finite set S whose number of elements is not a multiple of 7. Prove that there is some element x of S which is a *fixed point* for G, i.e. $g \cdot x = x$ for all $g \in G$.

The set S is partitioned into disjoint orbits by the action of G. By the orbit-stabilizer theorem, the number of elements in each orbit is a divisor of 49. The divisors of 49 are 1,7 and 49. Since the sum of the numbers of elements in the different orbits is not a multiple of 7, at least one of these numbers must be 1, which means that there is an element of S that is fixed by every element of G.

8. Let $GL(2,\mathbb{Z})$ denote the group of all 2×2 matrices with integer entries and with determinant 1 or -1, and let $\mathbb{Z}^2$ denote the set of all column vectors of length 2 with integer entries. So $GL(2,\mathbb{Z})$ acts on $\mathbb{Z}^2$ by $A \cdot \binom{x}{y} = A\binom{x}{y}$, for a matrix A in $GL(2,\mathbb{Z})$.

(a) What is the stabilizer in $\mathrm{GL}(2,\mathbb{Z})$ of $\binom{1}{0}$?

The set of all matrices of the form $\begin{pmatrix} 1 & a \\ 0 & \pm 1 \end{pmatrix}$, where $a \in \mathbb{Z}$.

(b) Show that the orbit of $\binom{1}{0}$ under this action is the set of all vectors of the form $\binom{a}{b}$ where a and b are integers with $\gcd(a,b) = 1$.

The orbit of $\binom{1}{0}$ is the set of all first columns of matrices in $\mathrm{GL}(2,\mathbb{Z})$, i.e. every vector of the form $\binom{a}{b}$ for which there exist integers c, d with $ad - bc = \pm 1$. Such integers c and d exist if and only if $\gcd(a,b) = 1$ (this is a consequence of the Euclidean Algorithm).

(c) What is the orbit of $\binom{2}{4}$?

By the above reasoning, $\binom{2}{4}$ does not belong to the orbit of $\binom{1}{0}$, but $\binom{1}{2}$ does. Thus $\binom{2}{4}$ belongs to the orbit of $\binom{2}{0}$, which is the set of all vectors of the from $\binom{a}{b}$ where $\gcd(a,b) = 2$.