

MA3343: GROUPS
SEMESTER 1 2018-19
PROBLEM SHEET 2 - OUTLINE SOLUTIONS

1. (a) Let D_6 be the group of symmetries of an equilateral triangle. Show that D_6 can be generated by one rotation and one reflection, or by two reflections.

- *One reflection and one rotation:* Choose a reflection (for example the reflection S_L in the vertical line L , and a rotation, for example the rotation R_{120} through 120° . The powers of R_{120} include all three rotations, and the other two reflections can be obtained by composing S_L with R_{120} in both orders. So by starting with just R_{120} and S_L we can obtain all six elements of D_6 .
- *Two reflections:* Start with two reflections in two different lines. By composing these we can get a rotation through 120° or 240° . The powers of this rotation include all three rotations, and we can obtain the missing reflection by composing a rotation with one of the reflections that we started with. So again we can generate all six elements from the two reflections that we start with.

- (b) Is it true that the group D_8 of symmetries of the square can be generated by *any* two reflections?

No: Start with the two reflections in the diagonals. These generate a subgroup of D_8 of order 4.

2. Prove that every group of order 7 is cyclic.

Proof. Let G be a group of order 7 and let x be a non-identity element of G . Let H be the cyclic subgroup of G generated by x . Then $|H| > 1$ since $x \neq \text{id}_G$. Then $|H| = 7$ by Lagrange's Theorem. Thus $H = G$ and G is cyclic. The same argument applies to any group of prime order. \square

3. Using Problem 2 (if you wish), prove that every group of order less than 6 is abelian.

Proof. Every group of order 1 is abelian obviously. Because 2,3 and 5 are prime, groups of these orders are cyclic and hence abelian by Problem 2. Now suppose that G is a group of order 4, and let a, b, c be the non-identity elements of G . Each of these generates a cyclic subgroup of order 2 or 4. If any of them has order 4, then G is cyclic and hence abelian. The alternative is that the cyclic subgroups generated by a, b and c all have order 2, which means that $a^2 = b^2 = c^2 = \text{id}$. In this case ab cannot be equal to id or to a or b and so must be equal to c . The same applies to ba . Similarly $ac = ca = b$ and $bc = cb = a$. Since all pairs of elements commute with each other G must be abelian. \square

4. (2,5) Let H be a subgroup of a finite group G , and let $x, y \in G$. Prove that x and y belong to the same left coset of H in G if and only if $x^{-1}y \in H$.

The elements x and y belong to the same left coset of H if and only if y belongs to the coset xH , which means $y = xh$ for some $h \in H$, or equivalently $x^{-1}y = h$ for some $h \in H$.

5. (1,2,5) Let $G = GL(2, \mathbb{Q})$, the group of nonsingular 2×2 matrices with rational entries, under matrix multiplication.

- (a) Let H be the subset of G consisting of all matrices in G with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the first column. Show that H is a subgroup of G . How can you tell by glancing at a pair of elements of G whether they belong to the same left coset of H ?

The matrices A and B are in the same left coset of H if $B = A \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$ for some $a, b \in \mathbb{Q}$ with $b \neq 0$. This means exactly that A and B have the same first column.

- (b) For $A \in G$, the *right coset* of H determined by A is the set of all matrices of the form BA where $B \in H$. How can you tell by glancing at a pair of elements of G whether they belong to the same right coset of H ?

The matrices A and B are in the same right coset of H if $B = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} A$ for some $a, b \in \mathbb{Q}$ with $b \neq 0$. This means that the second row of B is a non-zero scalar multiple of the second row of A , and that the difference between the first rows of A and B is a scalar multiple of the second row of either of them.

(c) Give an example of

- i. A pair of elements of G that belong to the same left coset of H and to the same right coset of H .
 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (both elements of H).
- ii. A pair of elements of G that belong to the same left coset of H but to different right cosets of H .
 $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.
- iii. A pair of elements of G that belong to different left cosets of H but to the same right coset of H .
 $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}$.
- iv. A pair of elements of G that belong to different left cosets of H and to different right cosets of H .
 $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$.

6. Let G be a group and let $x \in G$.

Definition: The *centralizer* of x in G , denoted $C_G(x)$, is defined to be the set of elements of G that commute with x , i.e.

$$C_G(x) = \{y \in G : yx = xy\}.$$

(a) i. What is the centre of Q ?

Answer: $\{1, -1\}$

ii. What is the centralizer in Q of the element -1 ?

Answer: All of Q .

iii. What is the centralizer in Q of the element i ?

Answer: $\{1, -1, i, -i\}$

(b) As usual let $GL(2, \mathbb{R})$ denote the group of invertible 2×2 matrices with entries in \mathbb{R} , under the operation of matrix multiplication.

i. What is the centralizer in $GL(2, \mathbb{R})$ of the diagonal matrix with entries $2, 3$ (in that order) along its main diagonal?

Answer: The set of invertible diagonal matrices.

ii. What is the centralizer in $GL(2, \mathbb{R})$ of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$?

Answer: The set of matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where $a, b \in \mathbb{R}$ and $a \neq 0$.

7. Let G be a group and let $x \in G$. Prove that $C_G(x)$, the centralizer of $x \in G$, is a subgroup of G .

Proof. First observe that $\text{id}_G x = x \text{id}_G = x$ by definition. So id_G commutes with x and $\text{id}_G \in C_G(x)$. Now suppose that $a, b \in C_G(x)$. Then $ax = xa$ and $bx = xb$. Hence

$$(ab)x = abx = a(bx) = a(xb) = (ax)b = (xa)b = x(ab),$$

and so $ab \in C_G(x)$ also. Thus $C_G(x)$ is closed under the group operation in G .

Finally suppose $a \in C_G(x)$. Then $ax = xa$ and so

$$a^{-1}axa^{-1} = a^{-1}xaa^{-1} \implies xa^{-1} = a^{-1}x.$$

Thus a^{-1} commutes with x and $a^{-1} \in C_G(x)$.

Since $C_G(x)$ contains the identity element of G , is closed under the operation of G , and contains the inverse of each of its elements, it is a subgroup of G . \square

8. **(2,4,5,6)** Let G be a finite group. Prove that the index of $Z(G)$ in G cannot be a prime number.

If G is abelian, then $Z(G) = G$ and $[G : Z(G)] = 1$. So suppose that G is not abelian, and let x be an element of G that does not belong to the centre of G . Then the centralizer of x in G is a subgroup of G that strictly contains $Z(G)$ (since it contains both $Z(G)$ and x) and which is strictly contained in G , since x does not commute with all elements of G . It follows that $C_G(x)$ is a subgroup of G that is intermediate between $Z(G)$ and G . From Lagrange's Theorem we can conclude that $[G : Z(G)]$ is not prime, since

$$\frac{|G|}{|Z(G)|} = \frac{|G|}{|C_G(x)|} \times \frac{|C_G(x)|}{|Z(G)|},$$

and both factors are integers greater than 1.