GRAPH THEORY (MA 522)

Problems to think about - Week 1

For discussion in Week 2 . . .

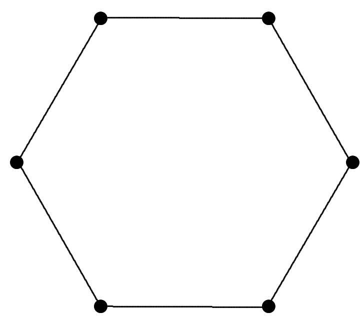
Note : Problems marked $\star$ are hard in my opinion. Problems marked with † are of some importance in the subject and should probably be discussed in next week's seminars. Exercises referenced are in Bondy and Murty's book.

1. (a) Let G be the graph obtained by deleting one edge from the complete bipartite graph $K_{3,3}$. Show that G is *planar* (i.e. can be drawn in the plane with no edges crossing).

   (b) Let H be the graph obtained by deleting one edge from the complete graph K_5. Show that H is planar.

   Note : The most obvious way to show that a graph is planar is to produce a picture of it without edge-crossings.

2. The cycle C_n on n vertices is the graph with n vertices and n edges forming a "cycle" of length n - for example C_6 is below. For what values of n is C_n a bipartite graph?



   Note : recall from the first seminar that what is special about a bipartite graph is that its vertex set falls into two parts, and that every edge connects a vertex from one part to a vertex from the other - there is no edge connecting two vertices in the same part. So a bipartite graph can be drawn with the vertices of one part on the left, the vertices of the other part on the right, and every edge going from left to right.

3. Suppose that G is a simple bipartite graph with n vertices. Show that the number of edges in G can be at most $n^2/4$. For which bipartite graphs is this maximum achieved?

   Note : if a bipartite graph on n vertices has k vertices in one part and $n-k$ in the other, how many edges can it have? When is this maximized?

4. † Suppose that G is a simple graph with at least two vertices. Show that the degrees of the vertices of G cannot all be different.

   Note : recall that the *degree* of a vertex is the number of edges with which that vertex is incident.

5. † Suppose that G is a graph with vertex set V, and let e be the number of edges in G.

   (a) Show that

$$\sum_{v \in V} \deg(v) = 2e.$$

   Hint : count the "ends of the edges" of G.

(b) Show that the number of vertices of odd degree in G must be even.

6. For each of the following lists of numbers, determine whether it can be the degree sequence of a simple graph.

   (a) $1, 1, 2, 2, 5, 5$
   (b) $1, 3, 3, 4, 5, 6, 6$
   (c) $2, 3, 3, 4, 5, 6, 6$
   (d) $2, 2, 3, 3, 4, 5, 6, 7$

   The problem of deciding whether a given list of natural numbers can be the degree sequence of a simple graph is not very easy - see Part (b) of Exercise 1.5.6 for a necessary and sufficient condition. However, for small examples like these we can get by with ad hoc methods. First, see if you can use part (b) of Problem 5 above. If not, think about what a graph with the proposed degree sequence could look like. For part (a) for example - if a simple graph on 6 vertices has two vertices of degree 5, what can they look like? Could there be any vertices of degree 1? In general for these problems thinking about the vertices of high degree and the vertices of low degree can sometimes be a way of getting going.

7. $\star$ Let $d_1, d_2, \ldots, d_k$ be any non–decreasing list of non–negative integers with the property that the sum of the d_i is even. Show that there is a (not necessarily simple) graph which has this list as its degree sequence.

   This is a difficult problem in my opinion because there is so little structure at our disposal. Try to describe a way of constructing a graph that will have the given sequence as its degree sequence - remember that your graph doesn't need to have any nice properties so don't make it more complicated than you have to. Hint : suppose the d_i were all even - what is the simplest thing you could do then? If they are not all even, then the number of vertices of odd degree is even - they could be grouped together in pairs if that would be helpful.

8. † Persuade yourself (whatever that means - you don't have to write out a proof) that if two graphs are isomorphic, they must have

   - The same number of vertices
   - The same number of edges
   - The same degree sequence

   If two simple graphs have all the above properties in common, must they be isomorphic? Investigate.

9. $\star$ (This is Exercise 1.2.13 - there is a typo in the book in the definition of *vertex–transitive*). Recall that an *automorphism* of a graph is a permutation of the vertices that preserves the relation of adjacency. A simple graph G is *vertex–transitive* if for every pair of vertices u and v of G there is an automorphism θ of G for which $\theta(u) = v$ (i.e. θ takes u to v).
   A simple graph G is *edge-transitive* if for every pair of edges uv and xy of G, there is an automorphism ϕ of G that takes the edge uv to the edge xy (here u, v, x and y denote vertices).
   Give an example of

   (a) A simple graph G that is vertex–transitive but not edge–transitive
   (b) A simple graph G that is edge–transitive but not vertex–transitive

   This problem has a $\star$ because the definitions might be a bit difficult to digest. Think about what a vertex transtive graph can look like - for example what its degree sequence can be.