

NUI GALWAY MATHEMATICS ENRICHMENT PROGRAMME

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The Plan: We will work in small groups on some of the following problems. The problems marked * are a bit harder. If you are taking part in this course for the first time, you might decide to ignore the * problems for now. If you were here last year and you have seen some of these problems before, you might like to have a look at the * problems. The problems marked † are important in the sense that they capture ideas that tend to arise quite often in mathematical problem solving.

1. †

- A drawer contains blue and black socks. How many socks must be taken from the drawer in order to be certain of having a pair (i.e. two socks of the same colour)?
- If the drawer contains white socks as well as blue and black ones, how many socks must be taken from it in order to be certain of having two socks of the same colour?
- How many socks must be taken from the drawer of part (a) in order to be certain of having *two* pairs?
- How many socks must be taken from the drawer of part (a) in order to be certain of having *two* pairs of the same colour (i.e. four socks all of the same colour)?

2. How many people must be assembled in order to be certain that

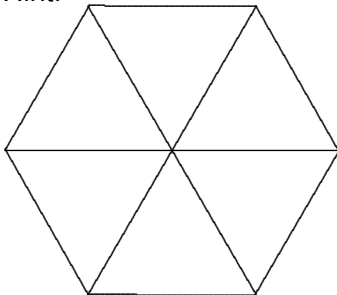
- Two have their birthdays in the same month?
- Three have their birthdays in the same month?
- Two have their birthdays on the same day?

3. 10 boxes contain a total of 50 pigeons. Every pair of boxes contains a total of at least 8 pigeons. What is the least number of pigeons that must be in any box?

Hint : Is it possible for a box to have no pigeons?

4. † A target has the shape of a regular hexagon with side length 1m. Hitting the target makes a mark. If the target is hit seven times, show that there are two marks at most 1 metre apart.

Hint:



5. * 101 distinct points are chosen in a 10 metre by 10 metre square. Prove that some three of these points form a triangle whose area is no more than 1m^2 .

6. †

- Take any three whole numbers. Show that among the three are two whose difference is an even number.
The *difference* of two numbers is what you get when you subtract the smaller from the greater.
- Take any six whole numbers. Show that among the six numbers are two whose difference is exactly divisible by 5.

Hint : If you are unsure what this problem is about or if you want to get a feeling for what is going on, try some examples. Give yourself a particular set of six whole numbers, and check to see if

some pair of your numbers differ by a multiple of 5. Is this an accident of your particular choice, or is it always going to happen? Change your choice of six numbers, and see if you can avoid the situation where two of your six numbers have a difference that is a multiple of 5. After a while you will probably start to suspect that it can't be avoided, then it is time to start looking for an explanation.

Think of the *remainders on division by 5*. For example the remainder on dividing 34 by 5 is 4, the remainder on dividing 187 by 5 is 2, and the remainder on dividing 85 by 5 is 0. How many different remainders are possible on division by 5? Can our six numbers all have different remainders on division by 5? If two numbers have the same remainder on division by 5, what can we say about their difference?

- (c) Take any twelve two-digit whole numbers. Show that among the twelve of them there must be two whose difference is a number of the form aa , i.e. the first and second digits in this difference are the same.

Hint : How would you describe two-digit numbers whose digits are the same?

7. * Choose any five positive integers. Show that from these five integers it is possible to select some (one or more) so that when you add together the integers in your selection, the number that you get is a multiple of five.

Hint : This is tricky. If one of the five numbers you start with is itself a multiple of five, then the problem is easy. If not, what can you say? Is there anything special about five? If you don't see what to do, could you try easier versions of this problem?

8. Choose any 12 positive integers. Show that amongst your 12 integers there are two whose difference is a multiple of 20, or there are two whose sum is a multiple of 20.

Hint : if two of your 12 integers have the same remainder on division by 20, then the difference of these two is a multiple of 20 so everything is ok. The case where the 12 integers all have different remainders on division by 20 is the one you have to think about. What about the sums of pairs then? Remember : *if you have $n + 1$ integers, they can't all have different remainders on division by n .*

9. * (Irish Mathematical Olympiad 2004) In a tennis tournament with 50 players, each player plays a match against each of the others. Every player wins at least one match. Prove that there must be three players A, B and C , so that A beat B , B beat C and C beat A .

10. † A *sequence* is an infinite list of objects, a list that never ends. For example we have the sequence of natural numbers which starts $1, 2, 3, \dots$ or the sequence that starts $1, 1, 1, \dots$ and continues with every term being a 1. But sequences do not necessarily have recognizable or repeating patterns.

- (a) Think about sequences in which every term is either 0 or 1. For example we could have the sequence $1, 0, 1, 0, 1, 0, \dots$ that continues with terms alternating between 0 and 1. We will say that two such sequences agree in position i , if they both have the same i th term. For example a sequence starting $0, 1, 1, 1, 0$ agrees with a sequence starting $1, 1, 1, 0, 0$ in the 2nd, 3rd and 5th positions but not in the 1st or 4th position.

Suppose you have three of these sequences with every term 0 or 1. Show that among the three sequences are two which agree in infinitely many positions.

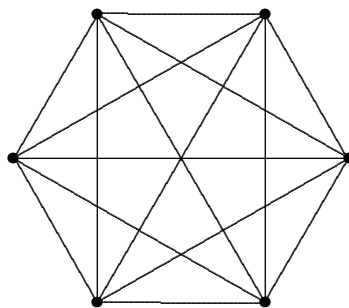
Note: One suggestion is to start by looking at the first two sequences. If they agree in infinitely many positions, then what we are asked to show is true. So suppose that they agree in only finitely many positions (maybe even none). In this case then there is a last position of agreement, beyond which the first two sequences disagree in all positions. What does this mean? Now introduce the third sequence. What can we say about this in relation to the first two sequences?

- (b) Now think about sequences in which every term is either 0, 1 or 2. Suppose you have four of these. Show that among the four are two which agree in infinitely many positions. Would this necessarily be true if we had three sequences instead of 4?

- (c) Take 11 real numbers between 0 and 1, and look at their decimal expansions. (If a number has a decimal expansion that terminates, think of it as ending in an infinite string of zeroes). Show

that among the 11 numbers there must be two whose decimal expansions agree in an infinite number of positions. Would this necessarily be true if we had 10 numbers instead of 11?

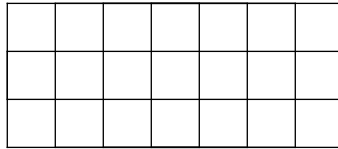
11. † Assume that given any two people, either they know each other (they are acquaintances) or they don't know each other (they are strangers).
- (a) Take any group of six people. Show that within the group of six, there are either three people all of whom know each other (3 mutual acquaintances) or three people none of whom know each other (3 mutual strangers).
- Hint : Represent the six people by six dots (vertices), all connected to each other by lines call *edges* as in the following picture.



If two of our six people are strangers, we will colour the edge joining them blue. If two are acquainted, we will colour the edge joining them red. So every edge will be coloured either red or blue. What we need to show is that no matter how we do this, we cannot avoid having either a group of three vertices joined to each other by three red edges (three mutual acquaintances) or a group of three vertices joined to each other by three blue edges (three mutual strangers). In order to get started, think of a single vertex A and look at the edges leading from A . Each of them is going to be coloured either red or blue. We cannot necessarily say anything about the number of red edges leading from A or the number of blue edges leading from A but we know that these two add up to 5. What can we say?

- (b) If you have five people instead of six, must there be either a group of three mutual acquaintances or a group of three mutual strangers?
- (c) In a group of 17 people, suppose that every pair is either a pair of friends, a pair of enemies or a pair of strangers. Show that the group must contain either three mutual friends, three mutual enemies or three mutual strangers.
12. * Show that in group of 16 people there is either a group of 4 mutual strangers or there is a person who is acquainted with 5 others. What about a group of 15 people?
13. In a party of 10 people, you are told that among any three people there are at least two who do not know each other. Prove that the party contains a group of four mutual strangers (i.e. a group of four people none of whom know each other).
14. Show that in a group of 9 people, there is a group of three mutual acquaintances or a group of four mutual strangers.
15. 33 buttons are placed on a 8×8 chequerboard, each occupying one square. Show that amongst the 33 buttons are five no two of which are in the same row or in the same column.

16. Each square in the 3×7 grid below is to be coloured either red or blue. Prove that no matter how this is done, there will be four squares of the same colour at the corners of a rectangle. Would this necessarily be true for a 3×6 grid?



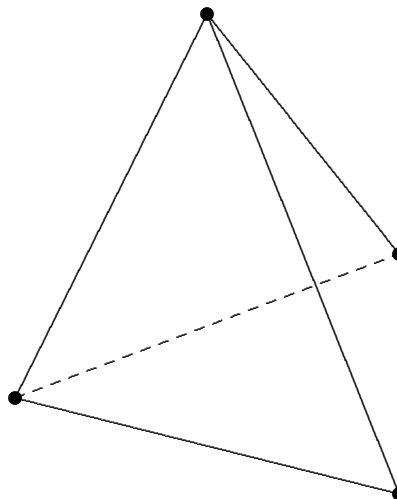
17. The points A , B and C form a triangle. The line L does not pass through any of the points A , B , C . Explain why L cannot pass through all three sides of the triangle.

Note: It may seem intuitively obvious from drawing pictures of triangles and lines that a line cannot intersect all three sides of a triangle unless it passes through one of the vertices. But saying that something is “obviously true” is not the same as proving it!

The following idea might help with this problem : the line L cuts the plane (that is, the whole region of space) into two parts, every point that does not belong to the line L is either on one side of L or on the other. In particular, each of the three points A , B , C is either on one side of the line L or on the other. If A and B are on different sides of L , does L cut the line segment AB ? What if A and B are on the same side of L ? What are the possibilities?

This problem is of a geometric flavour unlike the problems above. But think about how the reasoning needed to solve this problem is similar to that used for the earlier ones.

18. A *tetrahedron* is a three-dimensional object with four vertices (“corners”) and four triangular faces connected by six edges, as in the picture below.



Suppose that T is a tetrahedron and P is a plane that does not pass through any of the vertices of T . How many edges of T can P cut? Give all the possible answers.

19. (IrMO 1995) There are n^2 students in a class. Each week all the students participate in a table quiz. The teacher arranges the students into n teams of n players each. For as many weeks as possible, this is done in such a way that any pair of students who are members of the same team one week are not on the same team in subsequent week. Show that this cannot continue for $n + 2$ weeks.
20. (IrMO 1989) Each of the n members of a club is given a different item of information. They are allowed to share the information, but only in the following way : a pair may communicate by telephone, but during a telephone call only one person may speak. The person who speaks may tell the other person all the information (s)he knows. Determine the minimum number of telephone calls needed to convey all the information to each member.

Hint : if in doubt, try for specific values of n .