3.4 Further Properties of Eigenvectors and Eigenvalues

1. Suppose that v is an eigenvector of the square matrix A, corresponding to the eigenvalue λ. Then so is kv for any non-zero real number k. To see this note that

$$A(kv) = k\,A(v) = k(\lambda v) = (\lambda k)v = \lambda(kv).$$

2. If v is an eigenvector of A corresponding to the eigenvalue λ, then v is also an eigenvector of A^2 and the eigenvalue to which it corresponds is λ^2. To see this note

$$A^2(v) = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v.$$

   Similarly v is an eigenvector of A^n for any positive integer n, corresponding to the eigenvalue λ^n.

3. For any square matrix A, let A^T denote the transpose of A. Then $\det(A) = \det(A^T)$. It follows that

$$\det(\lambda I - A) = \det(\lambda I - A)^T = det(\lambda I - A^T).$$

   Thus A and A^T have the same characteristic equation, and they have the same eigenvalues. (However there is no general connection between the eigenvectors of A^T and those of A).

4. Suppose that A has the property that for each of its rows, the sum of the entries in that row is the same number s. For example if

$$A = \begin{pmatrix} 1 & 3 & 6 \\ 2 & -1 & 9 \\ -2 & 5 & 7 \end{pmatrix},$$

   the row sums of A are all equal to 10.

   Then

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} s \\ s \\ \vdots \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

   Thus the vector whose entries are all equal to 1 is an eigenvector of A corresponding to the eigenvalue s. In particular the common row sum s is an eigenvalue of A.

5. On the other hand suppose that the sum of the entries of every *column* of A is the same number k. Then by 4 above k is an eigenvalue of A^T and hence by 3 above k is an eigenvalue of A. In particular if the sum of the entries in every column of A is 1, then 1 is an eigenvalue of A.