3.3 Diagonalization

Let $A = \begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}$. Then $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are eigenvectors of A, with corresponding eigenvalues -2 and -6 respectively (check). This means

$$\begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -6\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} = \left(-2\begin{pmatrix} 1 \\ -2 \end{pmatrix} \; -6\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) = \begin{pmatrix} -2 & -6 \\ -4 & 12 \end{pmatrix}$$

We have

$$\begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -6 \end{pmatrix}$$

(Think about this). Thus $AE = ED$ where $E = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ has the eigenvectors of A as columns and $D = \begin{pmatrix} -2 & 0 \\ 0 & -6 \end{pmatrix}$ is the *diagonal* matrix having the eigenvalues of A on the main diagonal, in the order in which their corresponding eigenvectors appear as columns of E.

Definition 3.3.1 *A $n \times n$ matrix is A* diagonal *if all of its non-zero entries are located on its main diagonal, i.e. if $A_{ij} = 0$ whenever $i \neq j$.*

Diagonal matrices are particularly easy to handle computationally. If A and B are diagonal $n \times n$ matrices then the product AB is obtained from A and B by simply multiplying entries in corresponding positions along the diagonal, and $AB = BA$.
If A is a diagonal matrix and k is a positive integer, then A^k is obtained from A by replacing each entry on the main diagonal with its kth power.
Back to our Example : We have $AE = ED$. Note that $\det(E) \neq 0$ so E is invertible. Thus

$$\begin{aligned} AE &= ED \\ \Longrightarrow AEE^{-1} &= EDE^{-1} \\ \Longrightarrow A &= EDE^{-1}. \end{aligned}$$

It is convenient to write A in this form if for some reason we need to calculate powers of A. Note for example that

$$\begin{aligned} A^3 &= (EDE^{-1})(EDE^{-1})(EDE^{-1}) \\ &= EDI_2DI_2DE^{-1} \\ &= ED^3E^{-1} \\ &= E\begin{pmatrix} (-2)^3 & 0 \\ 0 & (-6)^3 \end{pmatrix}E^{-1}. \end{aligned}$$

In general $A^n = E\begin{pmatrix} (-2)^n & 0 \\ 0 & (-6)^n \end{pmatrix}E^{-1}$, for any positive integer n. (In fact this is true for negative integers too if we interpret A^{-n} to mean the nth power of the inverse A^{-1} of A).

Example 3.3.2 *Solve the recurrence relation*

$$\begin{aligned} x_{n+1} &= -4x_n + 1y_n \\ y_{n+1} &= 4x_n - 4y_n \end{aligned}$$

given that $x_0 = 1,\ y_0 = 1$.

Note: this means we have sequences $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ defined by the above relations. If for some n we know x_n and y_n, the relations tell us how to calculate x_{n+1} and y_{n+1}.

For example

$$\begin{aligned} x_1 &= -4x_0 + y_0 = -4(1) + 1 = -3 \\ y_1 &= 4x_0 - 4y_0 = 4(1) - 4(1) = 0 \end{aligned}$$

$$\begin{aligned} x_2 &= -4x_1 + y_1 = -4(-3) + 0 = 12 \\ y_2 &= 4x_1 - 4y_1 = 4(-3) - 4(0) = -12. \end{aligned}$$

Solution of the problem:

The relations can be written in matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} -4x_n + 1y_n \\ 4x_n - 4y_n \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix},$$

where A is the matrix $\begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}$. Thus

$$\begin{aligned}
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A \left(A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = A^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} &= A \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A \left(A^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = A^3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ etc.}
\end{aligned}$$

In general $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
To obtain general formulae for x_n and y_n we need a general formula for A^n. We have

$$A^n = (EDE^{-1})^n = ED^nE^{-1}$$

where $E = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} -2 & 0 \\ 0 & -6 \end{pmatrix}$.
Note

$$E^{-1} = -\frac{1}{4} \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}.$$

Thus

$$\begin{aligned}
A^n &= \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 \\ 0 & (-6)^n \end{pmatrix} \frac{1}{4} \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} (-2)^n & (-6)^n \\ 2(-2)^n & -2(-6)^n \end{pmatrix} \frac{1}{4} \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} (-2)^n(2) + (-6)^n(2) & (-2)^n - (-6)^n \\ 4(-2)^n - 4(-6)^n & 2(-2)^n + 2(-6)^n \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \frac{1}{4} \begin{pmatrix} (-2)^n(2) + (-6)^n(2) & (-2)^n - (-6)^n \\ 4(-2)^n - 4(-6)^n & 2(-2)^n + 2(-6)^n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 3(-2)^n + (-6)^n \\ 6(-2)^n - 2(-6)^n \end{pmatrix}
\end{aligned}$$

We conclude that

$$\begin{aligned} x_n &= \frac{3}{4}(-2)^n + \frac{1}{4}(-6)^n \\ y_n &= \frac{3}{2}(-2)^n - \frac{1}{2}(-6)^n \end{aligned}$$

for $n \geq 0$.
(This is easily verified for small values of n using the recurrence relations). See Problem Sheet 3 for more problems of this type.

Definition 3.3.3 *The $n \times n$ matrix A is* diagonalizable *(or diagonable) if there exists an invertible matrix E for which*

$$E^{-1}AE$$

is diagonal.

We have already seen that if E is a matrix whose columns are eigenvectors of A, then $AE = ED$, where D is the diagonal matrix whose entry in the (i,i) position is the eigenvalue of A to which the ith column of E corresponds as an eigenvector of A. If E is invertible then $E^{-1}AE = D$ and A is diagonalizable. Hence we have the following statement

1. If there exists an invertible matrix whose columns are eigenvectors of A, then A is diagonalizable.

On the other hand, suppose that A is diagonalizable. Then there exists an invertible $n \times n$ matrix E and a diagonal matrix D whose entry in the (i,i) position can be denoted d_i, for which

$$D = E^{-1}AE.$$

This means $ED = AE$, so

$$E \begin{pmatrix} d_1 & \dots & & \\ \vdots & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} = AE$$

. Looking at the jth column of each of these products shows that

$$\begin{pmatrix} E_{1j} \\ E_{2j} \\ \vdots \\ E_{nj} \end{pmatrix} d_j = A \begin{pmatrix} E_{1j} \\ E_{2j} \\ \vdots \\ E_{nj} \end{pmatrix}.$$

Thus the jth column of E is an eigenvector of A (with corresponding eigenvalue d_j). So

2. If the $n \times n$ matrix A is diagonalizable, then there exists an invertible matrix whose columns are eigenvectors of A.

Putting this together with 1. above gives

Theorem 3.3.4 *The square matrix A is diagonalizable if and only if there exists an invertible matrix having eigenvectors of A as columns.*

It is not true that every square matrix is diagonalizable.

Example 3.3.5 *Let* $A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$.

Then

$$\det(\lambda I - A) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

So $\lambda = 3$ is the only eigenvalue of A and it occurs twice.
Eigenvectors : Suppose $A\binom{x}{y} = 3\binom{x}{y}$. Then

$$\begin{matrix} 2x & - & y & = & 3x \\ x & + & 4y & = & 3y \end{matrix} \Longrightarrow x + y = 0, \quad x = -y.$$

So every eigenvector of A has the form $\begin{pmatrix} -y \\ y \end{pmatrix}$ for some non-zero real number y. Thus every 2×2 matrix having eigenvectors of A as columns as of the form $\begin{pmatrix} -a & -b \\ a & b \end{pmatrix}$ for some non-zero real numbers a and b. The determinant of such a matrix is $-ab - (-ab) = 0$. Thus no matrix having eigenvectors of A as columns is invertible, and A is not diagonalizable.

Although the above example shows that not all square matrices are diagonalizable, we do have the following fact.

Theorem 3.3.6 *Suppose that the* $n \times n$ *matrix* A *has* n distinct *eigenvalues* $\lambda_1, \ldots, \lambda_n$*. If* E *is a matrix whose columns are eigenvectors of* A *corresponding to the different eigenvalues, then* E *is invertible. Thus* A *is diagonalizable.*