3.2 The Characteristic Equation of a Matrix

Let A be a 2×2 matrix; for example

$$A = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix}.$$

If $\vec{v}$ is a vector in $\mathbb{R}^2$, e.g. $\vec{v} = [2, 3]$, then we can think of the components of $\vec{v}$ as the entries of a column vector (i.e. a 2×1 matrix). Thus

$$[2, 3] \leftrightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

If we multiply this vector on the left by the matrix A, we get another column vector with two entries :

$$A \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2(2) + 8(3) \\ 3(2) + (-3)(3) \end{pmatrix} = \begin{pmatrix} 28 \\ -3 \end{pmatrix}$$

So multiplication on the left by the 2×2 matrix A is a function sending the set of 2×1 column vectors to itself - or, if we wish, we can think of it as a function from the set of vectors in $\mathbb{R}^2$ to itself.

Note: In fact this function is an example of a *linear transformation* from $\mathbb{R}^2$ into itself. Linear transformations are functions which have certain interesting geometric properties. Basically they are functions which can be represented in this way by matrices.

In general, if v is a column vector with two entries, then Av is a another vector (with two entries), which typically does not resemble v at all. For example if $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ then

$$Av = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ -3 \end{pmatrix}$$

However, suppose $v = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$. Then

$$Av = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 8 \\ 3 \end{pmatrix} = \begin{pmatrix} 40 \\ 15 \end{pmatrix} = 5 \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

i.e. $A \begin{pmatrix} 8 \\ 3 \end{pmatrix} = 5 \begin{pmatrix} 8 \\ 3 \end{pmatrix}$, or

Multiplying the vector $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$ *(on the left) by the matrix* $\begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix}$ *is the same as multiplying it by* 5.

Terminology: $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$ is called an *eigenvector* for the matrix $A = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix}$ with corresponding *eigenvalue* 5.

Definition 3.2.1 *Let A be a $n \times n$ matrix, and let v be a non-zero column vector with n entries (so not all of the entries of v are zero). Then v is called an* eigenvector *for A if*

$$A\,v = \lambda\,v,$$

where λ is some real number.

In this situation λ is called an *eigenvalue* for A, and v is said to *correspond* to λ.

Note: "λ" is the symbol for the Greek letter *lambda*. It is conventional to use this symbol to denote an eigenvalue.

Example 3.2.2 *If* $A = \begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix}$ *and* $v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, *then*

$$Av = \begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -3v$$

Thus $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ *is an eigenvector for the matrix* $\begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix}$ *corresponding to the eigenvalue* -3.

Question: Given a $n \times n$ matrix A, how can we find its eigenvalues and eigenvectors?

Answer: We are looking for column vectors v and real numbers λ satisfying

$$\begin{aligned} Av &= \lambda v \\ \text{i.e. } \lambda v - Av &= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ \Longrightarrow \lambda I_n v - Av &= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ \Longrightarrow \underbrace{(\lambda I_n - A)}_{\text{a } n\times n \text{ matrix}} v &= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \end{aligned}$$

This may be regarded as a system of linear equations in which the coefficient matrix is $\lambda I_n - A$ and the variables are the n entries of the column vector v, which we can denote by $x_1, \dots, x_n$. We are looking for solutions to

$$(\lambda I_n - A) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

This system always has at least one solution : namely $x_1 = x_2 = \cdots = x_n = 0$ - all entries of v are zero. However this solution does *not* give an eigenvector since eigenvectors must be non-zero.

The system can have additional solutions only if $\det(\lambda I_n - A) = 0$ (otherwise if the square matrix $\lambda I_n - A$ is invertible, the system will have $x_1 = x_2 = \cdots = x_n = 0$ as its *unique* solution).

Conclusion: The *eigenvalues* of A are those values of λ for which $\det(\lambda I_n - A) = 0$.

Example 3.2.3 *Let* $A = \begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix}$. *Find all eigenvalues of* A *and find an eigenvector corresponding to each eigenvalue.*

Solution: We need to find all values of λ for which $\det(\lambda I_2 - A) = 0$.

$$\begin{aligned}\lambda I_2 - A &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda - 10 & 8 \\ -4 & \lambda + 2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\det(\lambda I_2 - A) &= (\lambda - 10)(\lambda + 2) - 8(-4) \\ &= \lambda^2 - 10\lambda + 2\lambda - 20 + 32 \\ &= \lambda^2 - 8\lambda + 12\end{aligned}$$

So $\det(\lambda I_2 - A)$ is a polynomial of degree 2 in λ. The eigenvalues of A are those values of λ for which

$$\det(\lambda I_2 - A) = 0$$

i.e. $\lambda^2 - 8\lambda + 12 = 0 \Longrightarrow (\lambda - 6)(\lambda - 2) = 0, \quad \lambda = 6$ or $\lambda = 2$

Eigenvalues of A : 6, 2.

To find an eigenvector of A corresponding to $\lambda = 6$, we need a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ for which

$$\begin{aligned}A \begin{pmatrix} x \\ y \end{pmatrix} &= 6 \begin{pmatrix} x \\ y \end{pmatrix} \\ \text{i.e.} \begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= 6 \begin{pmatrix} x \\ y \end{pmatrix} \\ \Longrightarrow \begin{pmatrix} 10x - 8y \\ 4x - 2y \end{pmatrix} &= \begin{pmatrix} 6x \\ 6y \end{pmatrix}\end{aligned}$$

$$\Longrightarrow 10x - 8y = 6x \quad \text{and} \quad 4x - 2y = 6y$$

Both of these equations say $x - 2y = 0$; hence *any* non-zero vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in which $x = 2y$ is an eigenvector for A corresponding to the eigenvalue 6. For example we can take $y = 1,\ x = 2$ to obtain the eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Exercises:

1. Show that $\begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 6\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
2. Find an eigenvector for A corresponding to the other eigenvalue $\lambda = 2$.

Definition 3.2.4 *Let A be a square matrix ($n \times n$). The* characteristic polynomial *of A is the determinant of the $n \times n$ matrix $\lambda I_n - A$. This is a polynomial of degree n in λ.*

Example 3.2.5

(a) Let $A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$. Then

$$\lambda I_2 - A = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \lambda - 4 & 1 \\ -2 & \lambda - 1 \end{pmatrix}$$

$\det(\lambda I_2 - A) = (\lambda - 4)(\lambda - 1) - 1(-2) = \lambda^2 - 5\lambda + 6$

Characteristic Polynomial of A: $\lambda^2 - 5\lambda + 6$.

(b) Let $B = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}$.

$$\lambda I_3 - B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} \lambda - 5 & -6 & -2 \\ 0 & \lambda + 1 & 8 \\ -1 & 0 & \lambda + 2 \end{pmatrix}$$

We can calculate $\det(\lambda I_3 - B)$ using cofactor expansion along the first row.

$$\begin{aligned} \det(\lambda I_3 - B) &= (\lambda - 5)[(\lambda + 1)(\lambda + 2) - (0)(8)] \\ &\qquad -(-6)[0(\lambda + 2) - 8(-1)] + (-2)[0(0) - (-1)(\lambda + 1)] \\ &= (\lambda - 5)(\lambda^2 + 3\lambda + 2) + 6(8) - 2(\lambda + 1) \\ &= \lambda^3 - 2\lambda^2 - 13\lambda - 10 + 48 - 2\lambda - 2 \\ &= \lambda^3 - 2\lambda^2 - 15\lambda + 36. \end{aligned}$$

<u>Characteristic polynomial of B</u> : $\lambda^3 - 2\lambda^2 - 15\lambda + 36$.

As we saw in Section 5.1, the eigenvalues of a matrix A are those values of λ for which $\det(\lambda I - A) = 0$; i.e., the eigenvalues of A are the *roots* of the characteristic polynomial.

Example 3.2.6 *Find the eigenvalues of the matrices A and B of Example 6.2.2.*

(a) $A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$

Characteristic Equation : $\lambda^2 - 5\lambda + 6 = 0 \Longrightarrow (\lambda - 3)(\lambda - 2) = 0$

<u>Eigenvalues of A</u>: $\lambda = 3, \ \lambda = 2$.

(b) $B = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$

Characteristic Equation: $\lambda^3 - 2\lambda^2 - 15\lambda + 36 = 0$

To find solutions to this equation we need to factor the characteristic polynomial, which is cubic in λ (in general solving a cubic equation like this is not an easy task unless we can factorize). First we try to find an integer root.

<u>Fact</u>: The only possible integer roots of a polynomial are factors of its constant term.

So in this example the only possible candidates for an integer root of the characteristic polynomial $p(\lambda) = \lambda^3 - 2\lambda^2 - 15\lambda + 36$ are the integer factors of 36 : i.e.

$$\pm 1, \ \pm 2, \ \pm 3, \ \pm 4, \ \pm 6, \ \pm 9, \ \pm 12, \ \pm 18, \ \pm 36$$

Try some of these :

$$\begin{aligned} p(1) &= 1^3 - 2(1)^2 - 15(1) + 36 \neq 0 \\ p(2) &= 2^3 - 2(2)^2 - 15(2) + 36 \neq 0 \\ p(3) &= 3^3 - 2(3)^2 - 15(3) + 36 = 0 \end{aligned}$$

$\Longrightarrow$ 3 is a root of $p(\lambda)$, and $(\lambda - 3)$ is a factor of $p(\lambda)$. Then

$$p(\lambda) = \lambda^3 - 2\lambda^2 - 15\lambda + 36 = (\lambda - 3)(\lambda^2 + a\lambda - 12)$$

To find a, look at the coefficients of λ^2 (or λ) on the left and right

$$\lambda^2 : \ -2 = -3 + a \Longrightarrow a = 1$$

$$\begin{aligned}\lambda^3-2\lambda^2-15\lambda+36 &= (\lambda-3)(\lambda^2+\lambda-12)\\ &= (\lambda-3)(\lambda-3)(\lambda+4)\\ &= (\lambda-3)^2(\lambda+4)\end{aligned}$$

Eigenvalues of B: $\lambda=3$ (occurring twice), $\lambda=-4$.

We conclude this section by calculating eigenvectors of B corresponding to these eigenvalues.

Example 3.2.7 *Let* $B=\begin{pmatrix}5&6&2\\0&-1&-8\\1&0&-2\end{pmatrix}$

From Example 3.2.5, the eigenvalues of B are $\lambda=3$ (occurring twice), $\lambda=-4$.

Find an eigenvector of B corresponding to the eigenvalue $\lambda=-4$.

Solution: We need a column vector $v=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, with entries not all zero, for which

$$\begin{pmatrix}5&6&2\\0&-1&-8\\1&0&-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=-4\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$\text{i.e.}\quad\begin{pmatrix}5x_1+6x_2+2x_3\\-x_2-8x_3\\x_1-2x_3\end{pmatrix}=\begin{pmatrix}-4x_1\\-4x_2\\-4x_3\end{pmatrix}$$

$$\Longrightarrow\begin{array}{rcl}5x_1+6x_2+2x_3&=&-4x_1\\-x_2-8x_3&=&-4x_2\\x_1-2x_3&=&-4x_3\end{array}\Longrightarrow\underbrace{\begin{array}{rcl}9x_1+6x_2+2x_3&=&0\\3x_2-8x_3&=&0\\x_1+2x_3&=&0\end{array}}_{\text{system of 3 equations in }x_1,x_2,x_3}$$

So we need to solve the system of linear equations with augmented matrix

$$\begin{pmatrix}9&6&2&0\\0&3&-8&0\\1&0&2&0\end{pmatrix}$$

Note: The coefficient matrix here is just $B-(-4)I_3$ i.e.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix} - \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

To find solutions to the system :

$$\begin{pmatrix} 9 & 6 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix} \quad \overset{R3 \leftrightarrow R1}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 9 & 6 & 2 & 0 \end{pmatrix} \quad \overset{R3 - 9 \times R1}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 6 & -16 & 0 \end{pmatrix} \quad \overset{R3 - 2 \times R2}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \overset{R2 \times \frac{1}{3}}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad : RREF$$

The variable x_3 is free : let $x_3 = t$. Then

$$\begin{aligned} x_1 + 2x_3 = 0 \quad &\Longrightarrow \quad x_1 = -2t \\ x_2 - \tfrac{8}{3}x_3 = 0 \quad &\Longrightarrow \quad x_2 = \tfrac{8}{3}t \end{aligned}$$

For example if we take $t = 3$ we find $x_1 = -6$ and $x_2 = 8$. Hence $v = \begin{pmatrix} -6 \\ 8 \\ 3 \end{pmatrix}$ is an eigenvector for B corresponding to $\lambda = -4$

Exercise: Check that $Bv = -4v$.

Notes:

1. To find an eigenvector v of a $n \times n$ matrix A corresponding to the eigenvalue λ : solve the system

$$(A - \lambda I_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

   i.e. the system whose coefficient matrix is $A - \lambda I_n$ and in which the constant term (on the right in each equation) is 0.

2. If v is an eigenvector of a square matrix A, corresponding to the eigenvalue λ, and if $k \neq 0$ is a real number, then kv is also an eigenvector of A corresponding to λ, since

$$A(kv) = k(Av) = k(\lambda v) = \lambda(kv)$$

In the above example any (non-zero) scalar multiple of $\begin{pmatrix} -6 \\ 8 \\ 3 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = -4$ (these arise from different choices of value for the free variable t in the solution of the relevant system of equations).

Example 3.2.8 *Find an eigenvector of B corresponding to the eigenvalue $\lambda = 3$.*

<u>Solution</u>: We need to solve the system whose augmented matrix consists of $B - 3I_3$ and a fourth column all of whose entries are zero.

$$B - 3I_3 = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix}$$

(obtained by subtracting 3 from each of the entries on the main diagonal of B and leaving the other entries unchanged).

We apply elementary row operations to the augmented matrix of the system :

$$\begin{pmatrix} 2 & 6 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 1 & 0 & -5 & 0 \end{pmatrix} \quad \begin{matrix} R1 \times \frac{1}{2} \\ \longrightarrow \\ R2 \times (-\frac{1}{4}) \end{matrix} \quad \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & -5 & 0 \end{pmatrix} \quad \begin{matrix} R3 - R1 \\ \longrightarrow \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{pmatrix} \quad \begin{matrix} R3 + 3 \times R2 \\ \longrightarrow \end{matrix} \quad \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R1 - 3 \times R2 \\ \longrightarrow \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad : RREF$$

Let $x_3 = t$. Then

$$\begin{aligned} x_1 - 5x_3 = 0 \quad &\Longrightarrow \quad x_1 = 5t \\ x_2 + 2x_3 = 0 \quad &\Longrightarrow \quad x_2 = -2t \end{aligned}$$

Eigenvectors are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5t \\ -2t \\ t \end{pmatrix}$$

for $t \in \mathbb{R},\ t \neq 0$. For example of we choose $t = 1$ we find that $v = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector for B corresponding to $\lambda = 3$. (Exercise: Check this).