Chapter 3

Eigenvalues and Eigenvectors

3.1 Powers of Matrices

Definition: Let A be a square matrix ($n \times n$). If k is a positive integer, then A^k denotes the matrix

$$\underbrace{A \times A \times \cdots \times A}_{k \text{ times}}.$$

Calculating matrix powers using the definition of matrix multiplication is computationally very laborious. One of the topics that we will discuss in this chapter is how powers of matrices may be calculated efficiently.

First we look at a reason for calculating such powers at all.

Example: Suppose that two competing Broadband companies, A and B, each currently have 50% of the market share. Suppose that over each year, A captures 10% of B's share of the market, and B captures 20% of A's share. What is each company's market share after 5 years?

Solution: Let a_n and b_n denote the proportion of the market held by A and N respectively at the end of the nth year. We have $a_0 = b_0 = 0.5$ (beginning of Year 1 = end of Year 0).

Now a_{n+1} and b_{n+1} depend on a_n and b_n according to

$$\begin{array}{rcrcr} a_{n+1} & = & 0.8a_n & + & 0.1b_n \\ b_{n+1} & = & 0.2a_n & + & 0.9b_n \end{array}$$

We can write this in matrix form as follows

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

We define $A = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$. Then

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = A \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = A \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \quad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = A \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = A^2 \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

In general

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = A^n \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}^n \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

So if we had an efficient way to calculate A^n, we could use it to calculate a_n and b_n.