2.4 A Method to Calculate the Inverse of a Matrix

Let $A = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{pmatrix}$.

Assume for now that A is invertible and suppose that

$$A^{-1} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Then $AA^{-1} + I_3$ and in particular the first column of AA^{-1} is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. This first column comes from the entries of A combined with the first column of A^{-1}. Thus we have

$$A \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This means $x = a_1,\ y = a_2,\ z = a_3$ is the unique solution of the system of linear equations given by

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the entries a_1, a_2, a_3 of the first column of A^{-1} will be written in the rightmost column of the RREF obtained from the matrix

$$\begin{pmatrix} 3 & 4 & -1 & 1 \\ 1 & 0 & 3 & 0 \\ 2 & 5 & -4 & 0 \end{pmatrix}.$$

Similarly the second and third columns of A^{-1} are respectively the unique solutions of the systems

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

They are respectively written in the rightmost columns of the RREFs obtained by EROs from the augmented matrices

$$\begin{pmatrix} 3 & 4 & -1 & 0 \\ 1 & 0 & 3 & 1 \\ 2 & 5 & -4 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 4 & -1 & 0 \\ 1 & 0 & 3 & 0 \\ 2 & 5 & -4 & 1 \end{pmatrix}.$$

So to find A^{-1} we need to reduce these three augmented matrices to RREF. This can be done with a single series of EROs if we start with the 3×6 matrix

$$A' = \begin{pmatrix} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{pmatrix}.$$

Method : Reduce A' to RREF. If the RREF has I_3 in its first three columns, then columns 4,5,6 contain A^{-1}.

We proceed as follows.

$$\begin{pmatrix} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{pmatrix} \quad \overset{R1 \leftrightarrow R2}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} R2 \to R2 - 3R1 \\ \longrightarrow \\ R3 \to R3 - 2R1 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{pmatrix} \quad \overset{R3 \to R3 - R2}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{pmatrix}$$

$$\overset{R3 \leftrightarrow R2}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 4 & -10 & 1 & -3 & 0 \end{pmatrix} \quad \overset{R3 \leftrightarrow R3 - 4R2}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{pmatrix}$$

$$\overset{R3 \times (-\frac{1}{10})}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{pmatrix} \quad \overset{R1 \to R1 - 3R3}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{pmatrix}$$

The above matrix is in RREF and its first three columns comprise I_3. We conclude that the matrix A^{-1} is written in the last three columns, i.e.

$$A^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{pmatrix}.$$

(It is easily checked that $AA^{-1} = I_3$).

Note The above procedure can be used to find the inverse of any $n \times n$ matrix A or to show that A is not invertible.

- Form the matrix $A' = (A|I_n)$.
- Apply elementary row operations to A' to reduce it to RREF.
- If a row having 0 in all of the first n positions appears, then A is not invertible.
- If the RREF has I_n in the first n columns, then the matrix formed by the last n columns is A^{-1}.

Example: If we apply this technique to the matrix $A = \begin{pmatrix} 3 & 2 & -5 \\ 1 & 1 & -2 \\ 5 & 3 & -8 \end{pmatrix}$ of Example 1.5.1, we get

$$\begin{pmatrix} 3 & 2 & -5 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 5 & 3 & -8 & 0 & 0 & 1 \end{pmatrix} \overset{R1 \leftrightarrow R2}{\longrightarrow} \begin{pmatrix} 1 & 1 & -2 & 0 & 1 & 0 \\ 3 & 2 & -5 & 1 & 0 & 0 \\ 5 & 3 & -8 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} R2 \to R2 - 3R1 \\ \longrightarrow \\ R3 \to R3 - 5R1 \end{matrix} \begin{pmatrix} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -3 & 0 \\ 0 & -2 & 2 & 0 & -5 & 1 \end{pmatrix} \overset{R3 \to R3 - 2R2}{\longrightarrow} \begin{pmatrix} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & * & * & * \end{pmatrix}$$

At this stage we can conclude that the matrix A is not invertible.

Example (MA203 Summer 2004 Q2 (a))

Find the last row of A^{-1} where

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 2 \\ -1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}.$$

Solution: Suppose that the last row of A^{-1} is $(x\ y\ z\ w)$. Then

$$(x\ y\ z\ w)\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 2 \\ -1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} = (0\ 0\ 0\ 1).$$

Thus

$$\begin{array}{ccccccccc} x & + & 2y & - & z & + & 2w & = & 0 \\ x & + & & & & & w & = & 0 \\ & & 2y & + & 2z & & & = & 0 \\ x & + & 2y & + & z & - & w & = & 1 \end{array}$$

So the entries of the fourth row of A^{-1} are the values of x, y, z, w in the unique solution of the system with augmented matrix

$$\begin{pmatrix} 1 & 2 & -1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 1 & 2 & 1 & -1 & 1 \end{pmatrix}.$$

The coefficient matrix here is A^T, the transpose of A. The right-hand column contains the entries of the fourth row of I_4. Applying elementary row operations to the above matrix results in

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{3}{7} \\ 0 & 1 & 0 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & 0 & -\frac{1}{7} \\ 0 & 0 & 0 & 1 & -\frac{3}{7} \end{pmatrix}.$$

We conclude that the final row of A^{-1} is

$$\left(\frac{3}{7}\ \frac{1}{7}\ \ -\frac{1}{7}\ \ -\frac{3}{7}\right).$$