2.3 The Inverse of a Matrix

Notation: For a positive integer n, we let $M_n(\mathbb{R})$ denote the set of $n \times n$ matrices with entries in $\mathbb{R}$.

Remark: When we work in the full set of matrices over $\mathbb{R}$, it is not always possible to add or multiply two matrices (these operations are subject to restrictions on the sizes of the matrices involved). However, if we restrict attention to $M_n(\mathbb{R})$ we can add any pair of matrices and multiply any pair of matrices, and we never move outside $M_n(\mathbb{R})$.

$M_n(\mathbb{R})$ is an example of the type of algebraic structure known as a *ring*.

In this section we will consider how we might define a version of "division" for matrices in $M_n(\mathbb{R})$.

In the set $\mathbb{R}$ of real numbers, dividing by a non-zero number x means multiplying by the *reciprocal* $1/x$ of x. For example if we divide a real number by 5 we are multiplying it by $\frac{1}{5}$: $\frac{1}{5}$ is the *reciprocal* or *multiplicative inverse* of 5 in $\mathbb{R}$. This means

$$\frac{1}{5} \times 5 = 1,$$

i.e., if you multiply 5 by $\frac{1}{5}$, you get 1; multiplying by $\frac{1}{5}$ "reverses" the work of multiplying by 5.

Definition 2.3.1 *Let A be a $n \times n$ matrix.If B is a $n \times n$ matrix for which*

$$AB = I_n \quad \text{and} \quad BA = I_n$$

then B is called an inverse *for A.*

Example: Let $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and let $B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$BA = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

So B is an inverse for A.

Remarks:

1. Suppose B and C are *both* inverses for a particular matrix A, i.e.

$$BA = AB = I_n \quad \text{and} \quad CA = AC = I_n$$

   Then

$$\begin{aligned} (BA)C &= I_nC = C \\ \text{Also } (BA)C &= B(AC) = BI_n = B \end{aligned}$$

   Hence $B = C$, and if A has an inverse, its inverse is unique. Thus we can talk about *the* inverse of a matrix.

2. The inverse of a $n \times n$ matrix A, if it exists, is denoted A^{-1}.

3. Not every square matrix has an inverse. For example the 2×2 zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does not.

In Example 1.5.1 we saw that the system

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array}$$

is inconsistent. This system can be written in matrix form as follows

$$\begin{pmatrix} 3x & + & 2y & - & 5z \\ x & + & y & - & 2z \\ 5x & + & 3y & - & 8z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}.$$

The left hand side of this equation can be written as the matrix product of the 3×3 coefficient matrix of the system and the column containing the variable names to obtain the following version :

$$\begin{pmatrix} 3 & 2 & -5 \\ 1 & 1 & -2 \\ 5 & 3 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$$

We let A denote the 3×3 matrix above.

If this matrix had an inverse, we could multiply both sides of the above equation on the left by A^{-1} to obtain

$$A^{-1}A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}.$$

This would mean that the system has a unique solution in which the values of x, y, z are the entries of the matrix $A^{-1}\begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$.

Since we know from Example 1.5.1 that the system has no solution, we must conclude that the matrix A has no inverse in $M_3(\mathbb{R})$.

General Fact : *Suppose that a system of equations ha a square coefficient matrix. If this coefficient matrix has an inverse the system has a unique solution.*

4. A square matrix that has an inverse is called invertible or non-singular. A matrix that has no inverse is called singular or non-invertible.

5. *A Converse to Item 3 above*: Suppose now that A is a $n \times n$ matrix (say 3×3) and that there is a system of equations with A as coefficient matrix that has a unique solution. Then the RREF obtained from the augmented matrix of the system has the following form

$$\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

Since the rightmost column does not contribute to the choice of elementary row operations, it follows that *every* system of linear equations having A as coefficient matrix has an augmented matrix with a RREF of the above form. Thus every system of equations having A as coefficient matrix has a unique solution.

In particular then the system described by

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

has a unique solution $x = a_1,\ y = a_2,\ z = a_3$.

Similarly the systems described by

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

have unique solutions given respectively by $x = b_1,\ y = b_2,\ z = b_3$ and $x = c_1,\ y = c_2,\ z = c_3$.

Now define

$$B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

and look at the product AB. This is the 3×3 identity matrix I_3. Thus B is an inverse for A and A is invertible. We conclude that if A is the coefficient matrix of a system having a unique solution, then A is invertible.

Putting this together with Item 3. above and the remarks at the end of Section 2.2, we obtain the following :

Theorem 2.3.2 *A $n \times n$ matrix A is invertible if and only if the following equivalent conditions hold.*

(a) Every system of linear equations with A as coefficient matrix has a unique solution.

(b) A can be reduced by elementary row operations to the $n \times n$ identity matrix.