2.2 The $n \times n$ Identity Matrix

Notation: The set of $n \times n$ matrices with real entries is denoted $M_n(\mathbb{R})$.

Example 2.2.1 $A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$ *and let* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Find* AI *and* IA.

Solution:

$$AI = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2(1)+3(0) & 2(0)+3(1) \\ -1(1)+2(0) & -1(0)+2(1) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = A$$

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1(2)+0(-1) & 1(3)+0(2) \\ 0(2)+1(-1) & 0(3)+1(2) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = A$$

Both AI and IA are equal to A : multiplying A by I (on the left or right) does not affect A.

In general, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is *any* 2×2 matrix, then

$$AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

and $IA = A$ also.

Definition 2.2.2 $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *is called the* 2×2 identity matrix *(sometimes denoted* I_2*).*

Remarks:

1. The matrix I behaves in $M_2(\mathbb{R})$ like the real number 1 behaves in $\mathbb{R}$ - multiplying a real number x by 1 has no effect on x.

2. Generally in algebra an *identity element* (sometimes called a *neutral element*) is one which has no effect with respect to a particular algebraic operation.

   For example 0 is the identity element for addition of numbers because adding zero to another number has no effect.

   Similarly 1 is the identity element for multiplication of numbers.

   I_2 is the identity element for multiplication of 2×2 matrices.

3. The 3×3 identity matrix is $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Check that if A is any 3×3 matrix then $AI_3 = I_3 A = A$.

Definition 2.2.3 *For any positive integer n, the $n \times n$* identity matrix I_n *is defined by*

$$I_n = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix}$$

(I_n has 1's along the "main diagonal" and zeroes elsewhere). The entries of I_n are given by :

$$(I_n)_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem 2.2.4 *1. If A is any matrix with n rows then $I_n A = A$.*

2. If A is any matrix with n columns, then $AI_n = A$.

(i.e. multiplying any matrix A (of admissible size) on the left or right by I_n leaves A unchanged).

Proof (of Statement 1 of the Theorem): Let A be a $n \times p$ matrix. Then certainly the product $I_n A$ is defined and its size is $n \times p$.

We need to show that for $1 \leq i \leq n$ and $1 \leq j \leq p$,the entry in the ith row and jth column of the product $I_n A$ is equal to the entry in the ith row and jth column of A.

$$\underset{I_n}{\begin{pmatrix} & & & 0 & \\ & & \ddots & \vdots & \\ 0 & \dots & 0 & 1 & 0 \\ & & & \vdots & \\ & & & 0 & \end{pmatrix}} \underset{A}{\begin{pmatrix} & A_{1j} & \\ & \vdots & \\ \dots & A_{ij} & \\ & \vdots & \\ & A_{nj} & \end{pmatrix}} = \underset{I_n A}{\begin{pmatrix} & & \vdots & \\ & & \vdots & \\ \dots & \dots & \bullet & \dots \\ & & \vdots & \end{pmatrix}}$$

$(I_n A)_{ij}$ comes from the ith row of I_n and the jth column of A.

$$\begin{aligned} (I_n A)_{ij} &= (0)(A)_{1j} + (0)(A)_{2j} + \cdots + (1)(A)_{ij} + \cdots + (0)(A)_{nj} \\ &= (1)(A)_{ij} \\ &= (A)_{ij} \end{aligned}$$

Thus $(I_n A)_{ij} = (A)_{ij}$ for all i and j - the matrices $I_n A$ and A have the same entries in each position. Then $I_n A = A$.

The proof of Statement 2 is similar. □