Chapter 2

Gauss-Jordan Elimination and Matrix Algebra

2.1 Review of Matrix Algebra

A $m \times n$ ("m by n") matrix is a matrix having m rows and n columns.

Example

$\begin{pmatrix} 2 & 3 & -1 \\ -3 & -4 & 0 \end{pmatrix}$ is a 2×3 matrix.

$\begin{pmatrix} 2 & 3 \\ 2 & 7 \\ 4 & 0 \end{pmatrix}$ is a 3×2 matrix.

Two matrices are said to have the same *size* if they have the same number of rows and the same number of columns. (So for example a 3×2 matrix and a 2×3 matrix are considered to be of different size).

Notation: If A is an $m \times n$ matrix, the entry appearing in the ith row and jth column of A (called the (i,j) position) is denoted $(A)_{ij}$.

Example: Let $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \end{pmatrix}$.

Then $(A)_{11} = 2,\ (A)_{21} = 4,\ (A)_{13} = -1$, etc.

Like numbers, matrices have arithmetic associated to them. In particular, a pair of matrices can be added or multiplied (subject to certain compatibility conditions on their sizes) to produce a new matrix.

Matrix Addition:

Let$A$ and B be matrices of the same size ($m \times n$). We define their *sum* $A+B$ to be the $m \times n$ matrix whose entries are given by

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$

Thus $A+B$ is obtained from A and B by adding entries in corresponding positions.

Example: Let $A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 1 & 2 & 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 & 0 & -2 \\ 3 & -3 & 1 & 1 \end{pmatrix}$. Then

$$A+B = \begin{pmatrix} 2+(-1) & 0+1 & -1+0 & -1+(-2) \\ 1+3 & 2+(-3) & 4+1 & 2+1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 4 & -1 & 5 & 3 \end{pmatrix}$$

Subtraction of matrices is now defined in the obvious way - e.g., with A and B as above, we have

$$A-B = \begin{pmatrix} 2-(-1) & 0-1 & -1-0 & -1-(-2) \\ 1-3 & 2-(-3) & 4-1 & 2-1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 & 1 \\ -2 & 5 & 3 & 1 \end{pmatrix}$$

Multiplication of a Matrix by a Real Number:

Let A be a $m \times n$ matrix and let c be a real number. Then cA is the $m \times n$ matrix with entries defined by

$$(cA)_{ij} = c(A)_{ij}$$

i.e. cA is obtained from A by multiplying every entry by c.

Example: If $A = \begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}$, then

$$2A = \begin{pmatrix} 4 & 2 \\ 6 & -8 \end{pmatrix}, \; -3A = \begin{pmatrix} -6 & -3 \\ -9 & 12 \end{pmatrix}, \; 0A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The $m \times n$ matrix whose entries are all zero is called the *zero* ($m \times n$) matrix.

Matrix Multiplication

Unlike addition, the manner in which matrices are multiplied does not appear completely natural at first glance.

Suppose that A is a $m \times p$ matrix and B is a $q \times n$ matrix. Then the product AB is defined if and only if $p = q$, i.e. if and only if

- The number of columns in A = the number of rows in B, or
- The number of entries in a row of A = the number of entries in a column of B.

In this case the size of AB is $m \times n$.

In general the following "cancellation law" holds for the size of matrix products:

$$\text{“}(m \times \not{p}) \times (\not{p} \times n) = m \times n\text{”}.$$

If A is a $m \times p$ matrix and B is a $p \times n$ matrix, then the product AB is defined and is a $m \times n$ matrix in which the entry in the ith row and jth column is given by combining the entries of the ith row of A with those of the jth column of B according to the following rule :

$$\text{product of 1st entries} + \text{product of 2nd entries} + \cdots + \text{product of } p\text{th entries}$$

Example 2.1.1 *Let* $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ *and let* $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$

Find AB *and* BA.

Solution :

1. $A: \ 2 \times 3, \ B: \ 3 \times 2 \Longrightarrow AB$ will be a 2×2 matrix.

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)(1)+3(0) & 2(1)+(-1)(-1)+3(2) \\ 1(3)+0(1)+(-1)(0) & 1(1)+0(-1)+(-1)(2) \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 9 \\ 3 & -1 \end{pmatrix}$$

2. $B: \ 3 \times 2, \ A: \ 2 \times 3 \Longrightarrow BA$ will be a 3×3.

$$BA = \begin{pmatrix} 7 & -3 & 8 \\ 1 & -1 & 4 \\ 2 & 0 & -2 \end{pmatrix}$$

(Exercise)

Note: $BA \neq AB$: Matrix multiplication is *not* commutative. In this example AB and BA are both defined but do not even have the same size. It is also possible for only one of AB and BA to be defined, for example this will happen if A is 2×4 and B is 4×3. Even if AB and BA are both defined and have the same size (for example if both are 3×3), the two products are typically different.

The next example shows how the computations involved in amtrix multiplication can arise sensibly.

Example 2.1.2 *A salesperson sells items of three types I, II, and III, costing €10, €20 and €30 respectively. The following table shows how many items of each type are sold on Monday morning and afternoon.*

	Type I	Type II	Type III
morning	3	4	1
afternoon	5	2	2

Let A denote the matrix

$$\begin{pmatrix} 3 & 4 & 1 \\ 5 & 2 & 2 \end{pmatrix}$$

Let B denote the 3×1 matrix whose entries are the prices of items of Type I, II and III respectively.

$$B = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}$$

Let C denote the 2×1 matrix whose entries are respectively the total income from morning sales and the total income from afternoon sales.Then

1st entry of C : $(3 \times 10) + (4 \times 20) + (1 \times 30) = 140$

1st entry of C : $(5 \times 10) + (2 \times 20) + (2 \times 30) = 150$

So $C = \begin{pmatrix} 140 \\ 150 \end{pmatrix}$

Now note that according to the definition of matrix multiplication we have

$$AB = C.$$

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1st entry of C: comes from combining the first row of A with the column of B according to :

$$\text{product of 1st entries} + \text{product of 2nd entries} + \text{product of 3rd entries}$$

$$(3 \times 10) + (4 \times 20) + (1 \times 30)$$

<u>2nd entry of C</u>: comes from combining the second row of A with the column of B in the same way.

$$(5 \times 10) + (2 \times 20) + (2 \times 30)$$