1.4 Leading Variables and Free Variables

Example 1.4.1 *Find the general solution of the following system :*

$$\begin{array}{rcrcrcrcrl} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 & \quad \text{I} \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 & \quad \text{II} \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 & \quad \text{III} \end{array}$$

Solution :

1. Write down the augmented matrix of the system :

$$\begin{array}{l} \text{Eqn I} \\ \text{Eqn II} \\ \text{Eqn III} \end{array} \quad \left(\begin{array}{rrrrr} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array}\right)$$
$$x_1 \quad x_2 \quad x_3 \quad x_4$$

   Note : This is the matrix of Example 1.2.4

2. Use Gauss-Jordan elimination to find a reduced row-echelon form from this augmented matrix. We have already done this in Examples 1.2.4 and 1.3.2 :-

$$\text{RREF :} \quad \left(\begin{array}{rrrrr} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$
$$x_1 \quad x_2 \quad x_3 \quad x_4$$

   This matrix corresponds to a new system of equations:

$$\begin{array}{rcrcrl} x_1 & + & 2x_4 & = & 4 & \quad \text{(A)} \\ x_2 & - & x_4 & = & 2 & \quad \text{(B)} \\ x_3 & + & x_4 & = & 2 & \quad \text{(C)} \end{array}$$

   Remark : The RREF involves 3 leading 1's, one in each of the columns corresponding to the variables x_1, x_2 and x_3. The column corresponding to x_4 contains no leading 1.

   Definition 1.4.2 *The variables whose columns in the RREF contain leading 1's are called* leading variables*. A variable whose column in the RREF does not contain a leading 1 is called a* free variable*.*

So in this example the leading variables are x_1, x_2 and x_3, and the variable x_4 is free. What does this distinction mean in terms of solutions of the system? The system corresponding to the RREF can be rewritten as follows :

$$\begin{array}{ccccll} x_1 & = & 4 & - & 2x_4 & \quad (A) \\ x_2 & = & 2 & + & x_4 & \quad (B) \\ x_3 & = & 2 & - & x_4 & \quad (C) \end{array}$$

i.e. this RREF tells us how the values of the leading variables x_1, x_2 and x_3 *depend* on that of the free variable x_4 in a solution of the system. In a solution, the free variable x_4 may assume the value of *any* real number. However, once a value for x_4 is chosen, values are immediately assigned to x_1, x_2 and x_3 by equations A, B and C above. For example

(a) Choosing $x_4 = 0$ gives $x_1 = 4 - 2(0) = 4,\ x_2 = 2 + (0) = 2,\ x_3 = 2 - (0) = 2$. Check that $x_1 = 4,\ x_2 = 2,\ x_3 = 2,\ x_4 = 0$ is a solution of the (original) system.

(b) Choosing $x_4 = 3$ gives $x_1 = 4 - 2(3) = -2,\ x_2 = 2 + (3) = 5,\ x_3 = 2 - (3) = -1$. Check that $x_1 = -2,\ x_2 = 5,\ x_3 = -1,\ x_4 = 3$ is a solution of the (original) system.

Different choices of value for x_4 will give different solutions of the system. The number of solutions is *infinite*.

The *general solution* is usually described by the following type of notation. We assign the *parameter* name t to the value of the variable x_4 in a solution (so t may assume any real number as its value). We then have

$$x_1 = 4 - 2t,\ x_2 = 2 + t,\ x_3 = 2 - t,\ x_4 = t;\ t \in \mathbb{R}$$

or

$$\textbf{General Solution :}\quad (x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t);\ t \in \mathbb{R}$$

This general solution describes the infinitely many solutions of the system : we get a *particular* solution by choosing a specific numerical value for t : this determines specific values for x_1, x_2, x_3 and x_4.

Example 1.4.3 *Solve the following system of linear equations :*

$$\begin{array}{rcrcrcrcrr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 & \text{I} \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 & \text{II} \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 & \text{III} \\ x_1 & - & x_2 & + & x_3 & - & x_4 & = & -6 & \text{IV} \end{array}$$

Remark : The first three equations of this system comprise the system of equations of Example 1.4.1. The problem becomes : Can we find a solution of the system of Example 1.4.1 which is in addition a solution of the equation $x_1 - x_2 + x_3 - x_4 = -6$?

<u>Solution</u> We know that every simultaneous solution of the first three equations has the form

$$x_1 = 4 - 2t,\ x_2 = 2 + t,\ x_3 = 2 - t,\ x_4 = t,$$

where t can be any real number . Is there some choice of t for which the solution of the first three equations is also a solution of the fourth? i.e. for which

$$x_1 - x_2 + x_3 - x_4 = -6 \text{ i.e. } (4 - 2t) - (2 + t) + (2 - t) - t = -6$$

Solving for t gives

$$\begin{aligned} 4 - 5t &= -6 \\ \Longrightarrow -5t &= 10 \\ \Longrightarrow t &= 2 \end{aligned}$$

$t = 2\ :\ x_1 = 4 - 2t = 4 - 2(2) = 0;\ x_2 = 2 + t = 2 + 2 = 4;\ x_3 = 2 - t = 2 - 2 = 0;\ x_4 = t = 2$

Solution : $x_1 = 0,\ x_2 = 4,\ x_3 = 0,\ x_4 = 2$ (or $(x_1, x_2, x_3, x_4) = (0, 4, 0, 2)$).

This is the *unique* solution to the system in Example 1.4.3.

<u>Remarks</u>:

1. To solve the system of Example 1.4.3 directly (without 1.4.1) we would write down the augmented matrix :

$$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \\ 1 & -1 & 1 & -1 & -6 \end{pmatrix}$$

<u>Check</u>: Gauss-Jordan elimination gives the reduced row-echelon form :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

which corresponds to the system

$$x_1 = 0;\ x_2 = 4;\ x_3 = 0;\ x_4 = 2$$

i.e. the unique solution is given exactly by the RREF. In this system, all four variables are leading variables. This is always the case for a system which has a unique solution : that each variable is a leading variable, i.e. corresponds in the RREF of the augmented matrix to a column which contains a leading 1.

2. The system of Example 1.4.1, consisting of Equations 1,2 and 3 of that in Example 1.4.3, had an infinite number of solutions. Adding the fourth equation in Example 1.4.3 pinpointed exactly one of these infinitely many solutions.