Chapter 1

Systems of Linear Equations

1.1 Introduction

Consider the equation

$$2x + y = 3.$$

This is an example of a *linear equation* in the variables x and y. As it stands, the statement "$2x + y = 3$" is neither true nor untrue : it is just a statement involving the abstract symbols x and y. However if we replace x and y with some particular pair of real numbers, the statement will become either true or false. For example

$$\begin{aligned} \text{Putting } & x = 1,\ y = 1 \text{ gives } 2x + y = 2(1) + (1) = 3 \text{ :True} \\ & x = 1,\ y = 2 \text{ gives } 2x + y = 2(1) + (2) \neq 3 \text{ :False} \\ & x = 0,\ y = 3 \text{ gives } 2x + y = 2(0) + (3) = 3 \text{ :True} \end{aligned}$$

Definition 1.1.1 *A pair* (x_0, y_0) *of real numbers is a* solution *to the equation* $2x + y = 3$ *if setting* $x = x_0$ *and* $y = y_0$ *makes the equation true; i,.e. if* $2x_0 + y_0 = 3$.

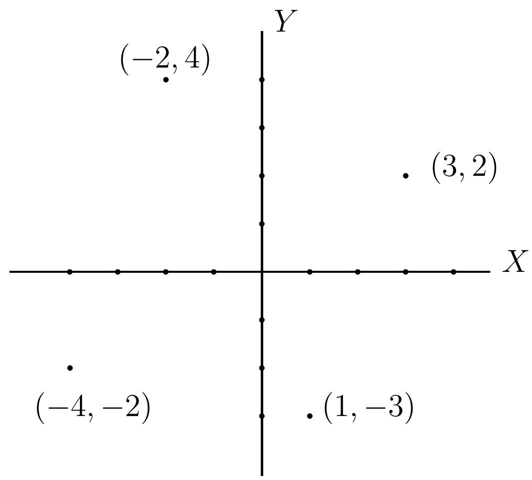
For example $(1, 1)$ and $(0, 3)$ are solutions - so are $(2, -1)$, $(3, -3)$, $(-1, 5)$ and $(-1/2, 4)$ (check these).

However $(1, 4)$ is not a solution since setting $x = 1$, $y = 4$ gives $2x + y = 2(1) + 4 \neq 3$.

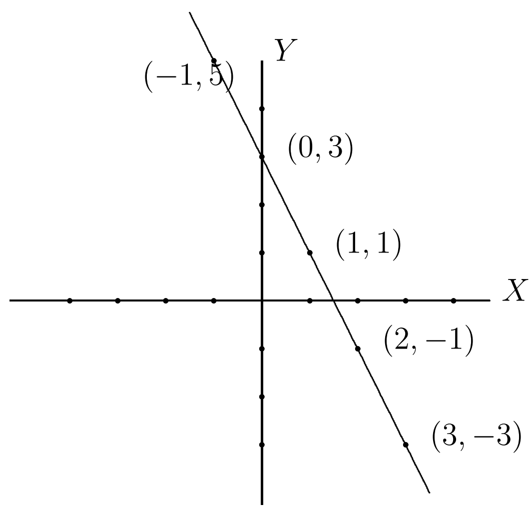
The set of all solutions to the equation is called its *solution set*.

Geometric Interpretation

Recall: The Cartesian Coordinate System. The 2-dimensional plane is described by a pair of perpendicular axes, labelled X and Y. A point is described by a pair of real numbers, its X and Y-coordinates.



We plot in the plane those points which correspond to the pairs of numbers which we found to be solutions to the equation $2x + y = 3$. These points form a *line* in the plane.



Now consider the equation $4x + 3y = 4$. Solutions to this equation include

$$(1, 0),\ (4, -4),\ (-2, 4),\ (-1/2, 2).$$

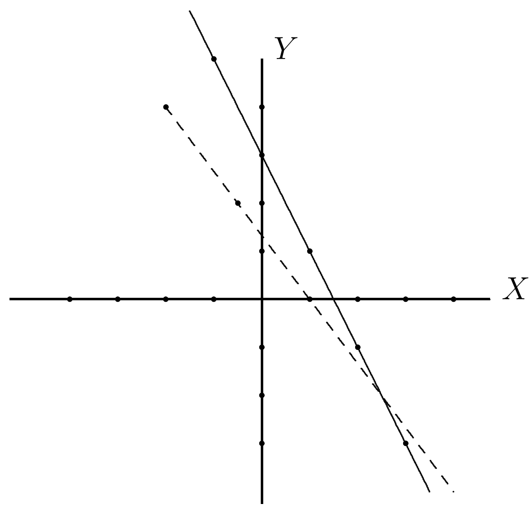
Again the full solution set forms a line.

Question: Consider the equations

$$2x + y = 3, \qquad 4x + 3y = 4$$

together. Can we find simultaneous solutions of these equations? This means - can we find pairs of numbers (x_0, y_0) such that setting $x = x_0$ and $y = y_0$ makes *both* equations true?

Equivalently - can we find a point of intersection of the two lines? From the picture it looks as if there is exactly one such point, at $(5/2, -2)$.



We can solve the problem algebraically as follows :

$$\left.\begin{array}{rcrcrl} 2x & + & y & = & 3 & \text{(A)} \\ 4x & + & 3y & = & 4 & \text{(B)} \end{array}\right\} \text{A system of linear equations.}$$

Step 1: Multiply Equation (A) by 2 : $4x + 2y = 6$ (A2).

Any solution of (A2) is a solution of (A).

Step 2: Multiply Equation (B) by -1 : $-4x - 3y = -4$ (B2)

Any solution of (B2) is a solution of (B).

Step 3: Now add equations (A2) and (B2).

$$\begin{array}{rcrcr} 4x & + & 2y & = & 6 \\ -4x & - & 3y & = & -4 \\ \hline \\ & & -y & = & 2 \end{array}$$

Step 4: So $y = -2$ and the value of y in any simultaneous solution of (A) and (B) is -2 : Now we can use (A) to find the value of x.

$$
\begin{aligned}
2x + y = 3 \text{ and } y = -2 \Longrightarrow 2x + (-2) &= 3 \\
\Longrightarrow 2x &= 5 \\
\Longrightarrow x &= \frac{5}{2}
\end{aligned}
$$

So $x = 5/2,\ y = -2$ is the *unique* solution to this system of linear equations.

This kind of "ad hoc" approach may not always work if we have a more complicated system, involving a greater number of variables, or more equations. We will devise a general strategy for solving complicated systems of linear equations.