MA203 Linear Algebra - Solutions to Homework 4

1. (H) The population of a region is divided into rural and urban residents. Each year, 5% of the urban residents move to rural areas and 15% of rural residents move to urban areas.

   (a) Write down the transition matrix of the Markov process describing this situation.

   Solution: $T = \begin{pmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{pmatrix}$.

   (b) If at the end of the year 2000, 60% of the population of the region were urban residents, what proportion of the population were urban residents at the end of 2002?

   Solution: Let $u_0 = 0.6$, $r_0 = 0.4$ denote the proportions of the population living in urban and rural areas respectively at the end of the year 2000. Where 2000 is designated as Year 0, let u_n and r_n respectively denote the proportions of the population living in urban and rural areas respectively at the end of Year n. Then

$$\begin{pmatrix} u_{n+1} \\ r_{n+1} \end{pmatrix} = \begin{pmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{pmatrix} \begin{pmatrix} u_n \\ r_n \end{pmatrix}$$

   Thus

$$\begin{pmatrix} u_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{pmatrix}^2 \begin{pmatrix} u_0 \\ r_0 \end{pmatrix} = \begin{pmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{pmatrix}^2 \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.654 \\ 0.346 \end{pmatrix}$$

   Thus at the end of 2002, 65.4% and 34.5% of the population are urban and rural residents respectively.

   (c) If this pattern of migration persists over many years, what proportion of the regions population will be rural residents in the long term?

   Solution: The steady state of this Markov process is the unique probability vector that is an eigenvector of T corresponding to the eigenvalue 1. If this vector is $\begin{pmatrix} x \\ y \end{pmatrix}$, we have

$$\begin{pmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow \begin{array}{rcl} 0.95x + 0.15y & = & x \\ 0.05x + 0.85y & = & y \end{array} \Longrightarrow x = 3y.$$

   In addition we require $x + y = 1$, hence $y = 0.25$, $x = 0.75$. Thus in the long term, 25% of the population will be rural residents.

4. (H) Consider a plant that can have red flowers (R), pink flowers (P), or white flowers (W), depending on the genotypes RR, RW and WW. When we cross each of genotypes with the genotype RW, we obtain the transition matrix

		Flowers of parent plant R	P	W
	R	0.5	0.25	0.0
Flowers of offspring plant	P	0.5	0.5	0.5
	W	0.0	0.25	0.5

Suppose that each successive generation is obtained by crossing only with plants of the RW genotype. When the process reaches equilibrium, what percentage of the plants will have red, pink, or white flowers?

Solution: The steady state vector for this Markov process is the unique probability vector that is an eigenvector of the above matrix T corresponding to the eigenvalue 1. Eigenvectors can be found by reducing the matrix $T - I_3$ to reduced row-echelon form. We obtain

$$\begin{pmatrix} -0.5 & 0.25 & 0.0 \\ 0.5 & -0.5 & 0.5 \\ 0.0 & 0.25 & -0.5 \end{pmatrix} \quad \overset{R3 \to R3 + (R1 + R2)}{\longrightarrow} \quad \begin{pmatrix} -0.5 & 0.25 & 0.0 \\ 0.5 & -0.5 & 0.5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{R1 \times (-2)}{\longrightarrow} \quad \begin{pmatrix} 1 & -0.5 & 0.0 \\ 0.5 & -0.5 & 0.5 \\ 0 & 0 & 0 \end{pmatrix} \quad \overset{R2 \to R2 - 0.5R1}{\longrightarrow} \quad \begin{pmatrix} 1 & -0.5 & 0.0 \\ 0 & -0.25 & 0.5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{R2 \times -4}{\longrightarrow} \quad \begin{pmatrix} 1 & -0.5 & 0.0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \overset{R1 \to R1 + 0.5R2}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ must satisfy $x = z$ and $y = 2z$. Since in addition $x + y + z = 1$ we have $4z = 1$, $z = 0.25$ and the steady state vector is $\begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix}$.

In the long term 25% of plants will have red flowers, 50% will have pink flowers, and 25% will have white flowers.