# MA203 Linear Algebra - Solutions to Homework 3

2. Use elementary row operations to calculate the determinant of each of the following matrices.

$$\text{(c) (H)} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 5 & 14 & -7 & 8 \\ -2 & 0 & -8 & -1 \\ -3 & 2 & -8 & 3 \end{pmatrix}$$

<u>Solution</u>: Use elementary row operations to reduce this matrix $A$ to upper triangular form.

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 5 & 14 & -7 & 8 \\ -2 & 0 & -8 & -1 \\ -3 & 2 & -8 & 3 \end{pmatrix} \begin{matrix} R2 \to R2 - 5R1 \\ R3 \to R3 + 2R1 \\ \longrightarrow \\ R4 \to R4 + 3R1 \end{matrix} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & -7 & 3 \\ 0 & 4 & -8 & 1 \\ 0 & 8 & -8 & 6 \end{pmatrix}$$

$$\begin{matrix} R3 \to R3 - R2 \\ \longrightarrow \\ R4 \to R4 - 2R2 \end{matrix} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & -7 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 6 & 0 \end{pmatrix} \begin{matrix} R4 \to R4 + 6R3 \\ \longrightarrow \end{matrix} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & -7 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -12 \end{pmatrix}$$

Call this upper triangular matrix $A'$. None of the EROs applied in converting $A$ to $A'$ changed the determinant, hence

$$\det(A) = \det(A') = 1 \times 4 \times (-1) \times (-12) = 48.$$

3. (H)

   (a) Calculate the third row of the inverse of the matrix of part (c) of Problem 2 above.
   <u>Solution</u>: The third row of the inverse of this matrix $A$ can be found as follows.

   - Write out the $4 \times 5$ matrix having $A^T$ in the first 4 columns and having the entries $0, 0, 1, 0$ in the fifth column.
   - Reduce this matrix to reduced row echelon form.
   - If the first four columns of the RREF comprise the $4 \times 4$ identity matrix, the entries of the fifth column are those of the third row of $A^{-1}$.

$$\begin{pmatrix} 1 & 5 & -2 & -3 & 0 \\ 2 & 14 & 0 & 2 & 0 \\ 0 & -7 & -8 & -8 & 1 \\ 1 & 8 & -1 & 3 & 0 \end{pmatrix} \quad \underset{R4 \to R4 - R1}{\overset{R2 \to R2 - 2R1}{\longrightarrow}} \quad \begin{pmatrix} 1 & 5 & -2 & -3 & 0 \\ 0 & 4 & 4 & 8 & 0 \\ 0 & -7 & -8 & -8 & 1 \\ 0 & 3 & 1 & 6 & 0 \end{pmatrix}$$

$$\overset{R2 \times 1/2}{\longrightarrow} \begin{pmatrix} 1 & 5 & -2 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & -7 & -8 & -8 & 1 \\ 0 & 3 & 1 & 6 & 0 \end{pmatrix} \quad \underset{R4 \to R4 - 3R2}{\overset{R3 \to R3 + 7R2}{\longrightarrow}} \quad \begin{pmatrix} 1 & 5 & -2 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 6 & 1 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix}$$

$$\overset{R3 \times -1}{\longrightarrow} \begin{pmatrix} 1 & 5 & -2 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -6 & -1 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix} \quad R4 \to R4 + 2R3 \longrightarrow \quad \begin{pmatrix} 1 & 5 & -2 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -6 & -1 \\ 0 & 0 & 0 & -12 & -2 \end{pmatrix}$$

$$\overset{R4 \times -1/12}{\longrightarrow} \begin{pmatrix} 1 & 5 & -2 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -6 & -1 \\ 0 & 0 & 0 & 1 & 1/6 \end{pmatrix} \quad \overset{R1 \to R1 - 5R2}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & -7 & -13 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -6 & -1 \\ 0 & 0 & 0 & 1 & 1/6 \end{pmatrix}$$

$$\underset{R2 \to R2 - R3}{\overset{R1 \to R1 + 7R3}{\longrightarrow}} \begin{pmatrix} 1 & 0 & 0 & -55 & -7 \\ 0 & 1 & 0 & 8 & 1 \\ 0 & 0 & 1 & -6 & -1 \\ 0 & 0 & 0 & 1 & 1/6 \end{pmatrix} \quad \underset{R3 \to R3 + 6R4}{\overset{\substack{R1 \to R1 + 55R4 \\ R2 \to R2 - 8R4}}{\longrightarrow}} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 13/6 \\ 0 & 1 & 0 & 0 & -1/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/6 \end{pmatrix}$$

We conclude that the third row of $A^{-1}$ is $\left(\frac{13}{6} \quad -\frac{1}{3} \quad 0 \quad \frac{1}{6}\right)$.

(b) Calculate the third column of the inverse of the matrix of part (c) of Problem 2 above.
<u>Solution</u>: We follow a similar approach to the above problem, reducing to RREF the matrix whose first four columns comprise $A$ and whose fourth column has entries $0, 0, 1, 0$. Omitting the details, we obtain

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 5 & 14 & -7 & 8 & 0 \\ -2 & 0 & -8 & -1 & 0 \\ -3 & 2 & -8 & 3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & 0 & 3/8 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/2 \end{pmatrix}.$$

The third column of $A^{-1}$ is $\begin{pmatrix} -1/4 \\ 3/8 \\ 0 \\ -1/2 \end{pmatrix}$.

5. (H) Let $B = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$.

(a) Find the characteristic polynomial of $B$.
<u>Solution</u>: The characteristic polynomial of $B$ is

$$\det(\lambda I - B) = \det \begin{pmatrix} \lambda - 3 & -5 \\ -4 & \lambda - 2 \end{pmatrix} = (\lambda - 3)(\lambda - 2) - (-5)(-4) = \lambda^2 - 5\lambda - 14.$$

(b) Find the eigenvalues of $B$.
<u>Solution</u>: $\lambda^2 - 5\lambda - 14 = 0 \Longrightarrow (\lambda - 7)(\lambda + 2) = 0$.
The eigenvalues of $B$ are 7 and $-2$.

(c) Find an eigenvector of $B$ corresponding to each eigenvalue.

- $\lambda = 7$:

$$B\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow \begin{array}{rcl} 3x + 5y & = & 7x \\ 4x + 2y & = & 7y \end{array}$$

Both equations say $4x = 5y$; any non-zero vector $\binom{x}{y}$ satisfying this is an eigenvector of $B$ corresponding to the eigenvalue 7. For example we can take $x = 5,\ y = 4$ to obtain the eigenvector $\binom{5}{4}$.

- $\lambda = -2$:

$$B\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow \begin{array}{rcl} 3x+5y & = & -2x \\ 4x+2y & = & -2y \end{array}$$

Both equations say $x = -y$; any non-zero vector $\binom{x}{y}$ satisfying this is an eigenvector of $B$ corresponding to the eigenvalue $-2$. For example we can take $x = -1,\ y = 1$ to obtain the eigenvector $\binom{-1}{1}$.

(d) Write down an invertible matrix $E$ and a diagonal matrix $D$ for which $B = EDE^{-1}$.

Solution: $E = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 7 & 0 \\ 0 & -2 \end{pmatrix}.$

(e) Calculate $B^4$.

Solution $B^4 = (EDE^{-1})^4 = ED^4E^{-1}$.

$$ED^4E^{-1} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 7^4 & 0 \\ 0 & (-2)^4 \end{pmatrix}\frac{1}{9}\begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 1341 & 1325 \\ 1060 & 1076 \end{pmatrix}$$

8. (H) Suppose that $x_0 = 1,\ y_0 = -1$ and for $n = 1, 2, \ldots$ the integers $x_n$ and $y_n$ are defined by

$$\begin{array}{rcl} x_n & = & 3x_{n-1} + 5y_{n-1} \\ y_n & = & 4x_{n-1} + 2y_{n-1}. \end{array}$$

Solve these recurrence relations to obtain explicit formulae for $x_n$ and $y_n$.

Solution: The recurrence relations can be written in matrix form as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}.$$

Let $B = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$; $B$ is the matrix of Problem 5 above. We have for $n \geq 1$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = B^n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = B^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

From Problem 5 we have

$$\begin{array}{rcl} B^n = ED^nE^{-1} & = & \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 7^n & 0 \\ 0 & (-2)^n \end{pmatrix}\frac{1}{9}\begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix} \\ & = & \frac{1}{9}\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 7^n & 7^n \\ -4(-2)^n & 5(-2)^n \end{pmatrix} \\ & = & \frac{1}{9}\begin{pmatrix} 5(7^n)+4(-2)^n & 5(7^n)-5(-2)^n \\ 4(7)^n-4(-2)^n & 4(7^n)+5(-2)^n \end{pmatrix} \end{array}$$

Then

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = B^n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = B^n\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9(-2)^n \\ -9(-2)^n \end{pmatrix}.$$

Thus $x_n = (-2)^n,\ y_n = -(-2)^n$.

Note : This could be seen more quickly by observing that $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $B$ corresponding to the eigenvalue $-2$, hence $B^n\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-2)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.