MA203 Linear Algebra 06/07 - Homework 3

Due date for problems marked (H) - Friday March 16

1. Use elementary row operations to find the inverses of the following matrices

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & 0 & 2 \\ 0 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & -2 \\ 2 & 1 & 0 \\ -2 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & -2 & 2 \\ 2 & -4 & 1 & 0 \\ 2 & 3 & 0 & -1 \end{pmatrix}.$$

2. Use elementary row operations to calculate the determinant of each of the following matrices.

(a) $\begin{pmatrix} 3 & 2 & 6 & 2 \\ 4 & 0 & 8 & 2 \\ 1 & 0 & 1 & -1 \\ 2 & 2 & 5 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -2 & 4 & -3 \\ -2 & 8 & -1 & -1 \\ 3 & -5 & 2 & 4 \\ 2 & -5 & 7 & 9 \end{pmatrix}$ (c) (H) $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 5 & 14 & -7 & 8 \\ -2 & 0 & -8 & -1 \\ -3 & 2 & -8 & 3 \end{pmatrix}$

3. (H)
   (a) Calculate the third row of the inverse of the matrix of part (c) of Problem 2 above.
   (b) Calculate the third column of the inverse of the matrix of part (c) of Problem 2 above.

4. Let $A = \begin{pmatrix} 3 & 2 \\ -3 & -4 \end{pmatrix}$.
   (a) Find the characteristic polynomial of A.
   (b) Find the eigenvalues of A.
   (c) Find an eigenvector of A corresponding to each eigenvalue.
   (d) Write down an invertible matrix E and a diagonal matrix D for which $A = EDE^{-1}$.
   (e) Calculate A^{10}.

5. (H) Let $B = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$.
   (a) Find the characteristic polynomial of B.
   (b) Find the eigenvalues of B.
   (c) Find an eigenvector of B corresponding to each eigenvalue.
   (d) Write down an invertible matrix E and a diagonal matrix D for which $B = EDE^{-1}$.
   (e) Calculate B^4.

6. Let $A = \begin{pmatrix} 0 & -3 & 5 \\ -4 & 4 & 10 \\ 0 & 0 & 4 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A.
   (b) Find all the eigenvalues of A.
   (c) Find an eigenvector corresponding to each eigenvalue.
   (d) Write down an invertible 3×3 matrix E for which $E^{-1}AE$ is diagonal.
   (e) Hence calculate A^5.

7. Let $A = \begin{pmatrix} 3 & 4 & -1 \\ -1 & -2 & 1 \\ 3 & 9 & 0 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A.
   (b) Find all the eigenvalues of A.
   (c) Find an eigenvector corresponding to each eigenvalue.
   (d) Explain why no invertible 3×3 matrix E for which $E^{-1}AE$ is diagonal exists.

8. (H) Suppose that $x_0 = 1,\ y_0 = -1$ and for $n = 1, 2, \ldots$ the integers x_n and y_n are defined by

$$\begin{aligned} x_n &= 3x_{n-1} + 5y_{n-1} \\ y_n &= 4x_{n-1} + 2y_{n-1}. \end{aligned}$$

   Solve these recurrence relations to obtain explicit formulae for x_n and y_n.

9. (H) The terms of the *Fibonacci Sequence* are given by $F_0 = 0,\ F_1 = 1$ and for $n \geq 0$ $F_{n+2} = F_{n+1} + F_n$. Thus each term is the sum of the previous two; the sequence begins $0, 1, 1, 2, 3, 5, 8, \ldots$

   (a) For $n = 0, 1, 2, 3$, verify directly that $F_n = \dfrac{1}{\sqrt{5}}(a^n - b^n)$, where $a = \frac{1}{2}(1 + \sqrt{5})$ and $b = \frac{1}{2}(1 - \sqrt{5})$.
   (b) By considering the relations

$$\begin{aligned} F_{n+2} &= 1F_{n+1} + 1F_n \\ F_{n+1} &= 1F_{n+1} + 0F_n \end{aligned}$$

   prove that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

   for all n.

   Hint: You will probably find this problem easier if you keep the names a and b for the numbers $\frac{1}{2}(1 + \sqrt{5})$ and $\frac{1}{2}(1 - \sqrt{5})$ until the very end. Note that these are the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Note also that $\binom{a}{1}$ and $\binom{b}{1}$ are eigenvectors of A. To confirm this you need to use the fact that $a^2 = a + 1$ and $b^2 = b + 1$, i.e. a and b are both roots of the characteristic polynomial of A.