MA203 Linear Algebra - Solutions to Homework 2

1. (H) Decide if each of the following statements is true or false. Explain your answer in each case.

   (a) Every system of linear equations in which the coefficient matrix is square has a unique solution.
   False - this is not true if the coefficient matrix is not invertible.

   (b) A system of equations with four equations and three variables can have a unique solution.
   True - the coefficient matrix of such a system is 4×3 - it has enough rows for every variable to be a leading variable.

   (c) A system of linear equations with three equations and four variables cannot have a unique solution.
   True - There are not enough rows in the coefficient matrix to accommodate four leading 1's, so not every variable can be a leading variable.

   (d) If A is a square matrix and there exists a system of equations having A as coefficient matrix and having a unique solution, then every system of linear equations having A as coefficient matrix has a unique solution.
   True - If there exists a system of equations having A as coefficient matrix and having a unique solution, then A is invertible.

   (e) A system of equations with three equations and four variables can be inconsistent.
   True - For example if the first equation says $x_1 = 0$ and the second says $x_1 = 1$, the system is inconsistent.

5. (H) Let $A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \\ -3 & 2 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ -2 & 3 \end{pmatrix}$.

   (a) Calculate the products AB and BC.
   Solution:

$$AB = \begin{pmatrix} 1 & 0 & 4 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \\ -3 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -11 & 8 & 1 \\ -6 & 5 & 4 \end{pmatrix}.$$

$$BC = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \\ -3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 4 & -9 \\ 1 & 5 \end{pmatrix}.$$

   (b) Verify that $(AB)C = A(BC)$.
   Solution

$$\begin{aligned} (AB)C &= \begin{pmatrix} -11 & 8 & 1 \\ -6 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 22 \\ -4 & 23 \end{pmatrix} \\ A(BC) &= \begin{pmatrix} 1 & 0 & 4 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & -9 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 22 \\ -4 & 23 \end{pmatrix} \end{aligned}$$

1. (H) Use Problem 6. above to find the unique solution to the folowing system of equations.

$$\begin{array}{rcrcrcr} 3x & - & 2y & + & z & = & 2 \\ 2x & - & y & + & 2z & = & 3 \\ x & - & 2y & - & 4z & = & 1 \end{array}$$

Solution: The system can be written as

$$\begin{pmatrix} 3 & -2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \text{ i.e. } A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = B\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 & -10 & -3 \\ 10 & -13 & -4 \\ -3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -17 \\ -23 \\ 7 \end{pmatrix}$$

Use Problem 6. above to find the unique solution to the folowing system of equations.

$$\begin{array}{rcrcrcr} 8x & - & 10y & - & 3z & = & 0 \\ 10x & - & 13y & - & 4z & = & -1 \\ -3x & + & 4y & + & z & = & 2 \end{array}$$

Solution: The system can be written as

$$\begin{pmatrix} 8 & -10 & -3 \\ 10 & -13 & -4 \\ -3 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \text{ i.e. } B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B^{-1}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = A\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix}$$

2. (H) Use elementary row operations to calculate the inverses of the following matrices, or to show that they are not invertible.

$$\text{(a) } A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix} \qquad \text{(b) } B = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$$

Solution: (a) Define $A' = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{pmatrix}$. Reduce A' to reduced row echelon form.

$$A' \overset{R1 \leftrightarrow R2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{pmatrix} \overset{R3 \to R3 - 4R1}{\longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{pmatrix}$$

$$\overset{R3 \to R3 + 3R2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{pmatrix} \overset{R3 \times (1/2)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$$

$$\underset{R2 \to R2 - 2R3}{\overset{R1 \to R1 - 3R3}{\longrightarrow}} \begin{pmatrix} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$$

Since the first three columns above comprise the 3×3 identity matrix, the second three comprise the inverse of the given matrix A. Thus

$$A^{-1} = \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}.$$

(b) Define $B' = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{pmatrix}$

Reduce B' to reduced row echelon form.

$$\begin{array}{c} R2 \to R2 + 3R1 \\ \longrightarrow \\ R3 \to R3 - 2R1 \end{array} \begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{pmatrix} \quad \begin{array}{c} R3 \to R3 + 3R2 \\ \longrightarrow \end{array} \begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{pmatrix}$$

$$\begin{array}{c} R3 \times (1/2) \\ \longrightarrow \end{array} \begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix} \quad \begin{array}{c} R1 \to R1 + 2R3 \\ \longrightarrow \\ R1 \to R2 + 2R3 \end{array} \begin{pmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Since the first three columns above comprise the 3×3 identity matrix, the second three comprise the inverse of the given matrix B. Thus

$$B^{-1} = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$