MA203 Linear Algebra - Solutions to Homework 1

2. (H) Consider the following system of equations :

$$\begin{array}{rcrcrcr} 2x & - & y & - & 3z & = & -8 \\ x & + & 2y & + & z & = & 1 \\ 4x & - & 4y & - & 5z & = & -2 \end{array}$$

(a) Write down the augmented matrix of this system.

$$\begin{pmatrix} 2 & -1 & -3 & -8 \\ 1 & 2 & 1 & 1 \\ 4 & -4 & -5 & -2 \end{pmatrix}$$

(b) Use elementary row operations to reduce this matrix to

(i) row-echelon form

$$\begin{pmatrix} 2 & -1 & -3 & -8 \\ 1 & 2 & 1 & 1 \\ 4 & -4 & -5 & -2 \end{pmatrix} \overset{R1 \leftrightarrow R2}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & -3 & -8 \\ 4 & -4 & -5 & -2 \end{pmatrix}$$

$$\underset{R3 \to R3 - 4R1}{\overset{R2 \to R2 - 2R1}{\longrightarrow}} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & -5 & -10 \\ 0 & -12 & -9 & -6 \end{pmatrix} \underset{R3 \times (-1/3)}{\overset{R2 \times (-1/5)}{\longrightarrow}} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 4 & 3 & 2 \end{pmatrix}$$

$$\overset{R3 \to R3 - 4R1}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -6 \end{pmatrix} \overset{R3 \times (-1)}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

(ii) *reduced* row-echelon form.

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 6 \end{pmatrix} \overset{R1 \to R1 - 2R2}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

$$\underset{R2 \to R2 - R3}{\overset{R1 \to R1 + R3}{\longrightarrow}} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

(c) Using your answer(s) to (b) above, find all solutions of the system of equations.
The system has a unique solution $x = 3,\ y = -4,\ z = 6$.

3. (H) Consider the following system of equations :

$$\begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & - & 2x_3 & + & x_4 & = & -8 \\ x_1 & - & 2x_2 & - & 2x_3 & + & 6x_4 & = & -3 \\ 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & -7 \end{array}$$

(a) Check that

(i) $x_1 = -1,\ x_2 = -1,\ x_3 = 2,\ x_4 = 0$

(ii) $x_1 = -11,\ x_2 = 14,\ x_3 = -3,\ x_4 = 5$

(iii) $x_1 = 3,\ x_2 = -7,\ x_3 = 4,\ x_4 = -2$

(iv) $x_1 = -101,\ x_2 = 149,\ x_3 = -48,\ x_4 = 50$

are all solutions of this system.

This is easily checked by simply substituting the values in each of (i), (ii), (iii), (iv) above into each of the three equations. The point of this exercise is to see that the system has many different solutions.

(b) Write down the augmented matrix of this system.

$$\begin{pmatrix} 3 & 1 & -2 & 1 & -8 \\ 1 & -2 & -2 & 6 & -3 \\ 2 & -1 & -3 & 4 & -7 \end{pmatrix}$$

(c) Use elementary row operations to reduce this matrix to

(i) row-echelon form

$$\begin{pmatrix} 3 & 1 & -2 & 1 & -8 \\ 1 & -2 & -2 & 6 & -3 \\ 2 & -1 & -3 & 4 & -7 \end{pmatrix} \quad \overset{R1 \leftrightarrow R2}{\longrightarrow} \quad \begin{pmatrix} 1 & -2 & -2 & 6 & -3 \\ 3 & 1 & -2 & 1 & -8 \\ 2 & -1 & -3 & 4 & -7 \end{pmatrix}$$

$$\begin{matrix} R2 \to R2 - 3R1 \\ \longrightarrow \\ R3 \to R3 - 2R1 \end{matrix} \quad \begin{pmatrix} 1 & -2 & -2 & 6 & -3 \\ 0 & 7 & 4 & -17 & 1 \\ 0 & 3 & 1 & -8 & -1 \end{pmatrix} \quad \overset{R2 \to R2 - 2R3}{\longrightarrow} \quad \begin{pmatrix} 1 & -2 & -2 & 6 & -3 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 3 & 1 & -8 & -1 \end{pmatrix}$$

$$\overset{R3 \to R3 - 3R2}{\longrightarrow} \quad \begin{pmatrix} 1 & -2 & -2 & 6 & -3 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & -5 & -5 & -10 \end{pmatrix} \quad R3 \times (-1/5) \longrightarrow \quad \begin{pmatrix} 1 & -2 & -2 & 6 & -3 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

(ii) *reduced* row-echelon form.

$$\begin{pmatrix} 1 & -2 & -2 & 6 & -3 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix} \quad R1 \to R1 + 2R2 \longrightarrow \quad \begin{pmatrix} 1 & 0 & 2 & 4 & 3 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

$$\begin{matrix} R1 \to R1 - 2R3 \\ \longrightarrow \\ R2 \to R2 - 2R3 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

(d) Using your answer to (ii) in (c) above, find the general solution of the system of equations.

Solutions satisfy

$$x_1 = -1 - 2x_4, \; x_2 = -1 + 3x_4, \; x_3 = 2 - x^4.$$

The variable x_4 is free - write $x_4 = t$. The general solution is given by

$$(x_1, x_2, x_3, x_4) = (-1 - 2t, -1 + 3t, 2 - t, t), \; t \in \mathbb{R}.$$

(e) Find all solutions of the following system of equations :

$$\begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & - & 2x_3 & + & x_4 & = & -8 \\ x_1 & - & 2x_2 & - & 2x_3 & + & 6x_4 & = & -3 \\ 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & -7 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 3 \end{array}$$

Since the first three equations coincide with the above system, we need all values of t for which the general solution to Equations 1,2,3 also satisfies Equation 4. Thus

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 3 \\ \Longrightarrow (-1 - 2t) + (-1 + 3t) + (2 - t) + t &= 3 \\ \Longrightarrow t &= 3 \end{aligned}$$

The unique solution is given by

$$(x_1, x_2, x_3, x_4) = (-7, 8, -1, 3).$$

4. (H) Consider the following system of equations :

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & + & x_3 & - & 2x_4 & = & -2 \\ x_1 & - & 2x_2 & - & 2x_3 & + & 3x_4 & = & -2 \\ 3x_1 & - & 6x_2 & - & 4x_3 & + & 2x_4 & = & -9 \end{array}$$

(a) Write down the augmented matrix of this system.

$$\begin{pmatrix} 2 & -4 & 1 & -2 & -2 \\ 1 & -2 & -2 & 3 & -2 \\ 3 & -6 & -4 & 2 & -9 \end{pmatrix}$$

(b) Use elementary row operations to reduce this matrix to

(i) row-echelon form

$$\begin{pmatrix} 2 & -4 & 1 & -2 & -2 \\ 1 & -2 & -2 & 3 & -2 \\ 3 & -6 & -4 & 2 & -9 \end{pmatrix} \overset{R1 \leftrightarrow R2}{\longrightarrow} \begin{pmatrix} 1 & -2 & -2 & 3 & -2 \\ 2 & -4 & 1 & -2 & -2 \\ 3 & -6 & -4 & 2 & -9 \end{pmatrix}$$

$$\begin{array}{c} R2 \to R2 - 2R1 \\ \longrightarrow \\ R3 \to R3 - 3R1 \end{array} \begin{pmatrix} 1 & -2 & -2 & 3 & -2 \\ 0 & 0 & 5 & -8 & 2 \\ 0 & 0 & 2 & -7 & -3 \end{pmatrix} \overset{R2 \to R2 - 2R3}{\longrightarrow} \begin{pmatrix} 1 & -2 & -2 & 3 & -2 \\ 0 & 0 & 1 & 6 & 8 \\ 0 & 0 & 2 & -7 & -3 \end{pmatrix}$$

$$\overset{R3 \to R3 - 2R2}{\longrightarrow} \begin{pmatrix} 1 & -2 & -2 & 3 & -2 \\ 0 & 0 & 1 & 6 & 8 \\ 0 & 0 & 0 & -19 & -19 \end{pmatrix} \overset{R3 \times (-1/19)}{\longrightarrow} \begin{pmatrix} 1 & -2 & -2 & 3 & -2 \\ 0 & 0 & 1 & 6 & 8 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(ii) *reduced* row-echelon form.

$$\begin{pmatrix} 1 & -2 & -2 & 3 & -2 \\ 0 & 0 & 1 & 6 & 8 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \overset{R1 \to R1 + 2R2}{\longrightarrow} \begin{pmatrix} 1 & -2 & 0 & 15 & 14 \\ 0 & 0 & 1 & 6 & 8 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{c} R1 \to R1 - 15R3 \\ \longrightarrow \\ R2 \to R2 - 6R3 \end{array} \begin{pmatrix} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(c) Using your answer to (ii) in (b) above, find the general solution of the system of equations.
From the RREF :

$$x_1 = -1 + 2x_2, \; x_3 = 2, \; x_4 = 1.$$

The variable x_2 is free. If we write $x_2 = t$, we have the general solution

$$(x_1, x_2, x_3, x_4) = (-1 + 2t, t, 2, 1), \; t \in \mathbb{R}.$$

(d) Show that the system

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & + & x_3 & - & 2x_4 & = & -2 \\ x_1 & - & 2x_2 & - & 2x_3 & + & 3x_4 & = & -2 \\ 3x_1 & - & 6x_2 & - & 4x_3 & + & 2x_4 & = & -9 \\ x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & 0 \end{array}$$

is inconsistent.
The first three equations comprise the above system. So any solution of this new system is a solution of the original system which also satisfies Equation 4. Suppose for some value of t that the general solution of the first three equations also satisfies the fourth. Then

$$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 &= 0 \\ \Longrightarrow -1 + 2t - 2t + 2 + 1 &= 0 \\ \Longrightarrow 2 &= 0 \end{aligned}$$

This equation cannot be satisfied (by any value of t), so no simultaneous solution of the first three equations also satisfies the fourth. So the system is inconsistent.