1.2 Linear Transformations of $\mathbb{R}^2$

When dealing with objects such as $\mathbb{R}^2$ and functions between them, it is common practice to look at functions that preserve algebraic structure.

Example 1.2.1 *(a) Consider the function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined for all $(x, y) \in \mathbb{R}^2$ by*

$$T(x, y) = (3x - 2y, -x).$$

So $T(2, 4) = (3(2) - 2(4), -2) = (-2, -2)$, etc.
Suppose that $u = (a_1, b_1)$ and $v = (a_2, b_2)$ are elements of $\mathbb{R}^2$. Then we can form $u + v = (a_1 + a_2, b_1 + b_2) \in \mathbb{R}^2$.

Question: Will adding u and v and then applying T to the result give the same outcome as applying T separately to u and v and then adding their images?
To check :

$$\begin{aligned}
T(u+v) &= T((a_1, b_1) + (a_2, b_2)) \\
&= T(a_1 + a_2, b_1 + b_2) \\
&= (3(a_1 + a_2) - 2(b_1 + b_2), -(a_1 + a_2)) \\
&= (3a_1 + 3a_2 - 2b_1 - 2b_2, -a_1 - a_2). \\
T(u) + T(v) &= T(a_1, b_1) + T(a_2, b_2) \\
&= (3a_1 - 2b_1, -a_1) + (3a_2 - 2b_2, -a_2) \\
&= (3a_1 - 2b_1 + 3a_2 - 2b_2, -a_1 - a_2) \\
&= (3a_1 + 3a_2 - 2b_1 - 2b_2, -a_1 - a_2) \\
&= T(u+v).
\end{aligned}$$

So $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^2$. We say that T is *additive* or that T respects addition.

(b) Another Question: Let $u = (a, b)$ in $\mathbb{R}^2$ and suppose $k \in \mathbb{R}$. Is multiplying u by k and then applying T the same as applying T to u and then multiplying the result by k?
To check :

$$\begin{aligned}
T(ku) &= T(ka, kb) \\
&= (3ka - 2kb, -ka). \\
kT(u) &= kT(a, b) \\
&= k(3a - 2b, -a) \\
&= (3ka - 2kb, -ka) \\
&= T(ku).
\end{aligned}$$

So $T(ku) = kT(u)$ for all $u \in \mathbb{R}^2$ and $k \in \mathbb{R}$. We say that T respects scalar multiplication.

(c) Suppose a function $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by

$$S(x, y) = (xy, -y)$$

for all $(x, y) \in \mathbb{R}^2$. Then S is not additive since for example

$$\begin{aligned}
S((1,0) + (0,2)) &= S(1,2) = (2, -2). \\
S(1,0) + S(0,2) &= (0,0) + (0,-2) = (0,-2) \neq (2,-2).
\end{aligned}$$

Nor does S respect scalar multiplication since for example

$$\begin{aligned} S(2(1,1)) &= S(2,2) = (4,-2) \\ \text{but } 2S(1,1) &= 2(1,-1) = (2,-2) \neq (4,-2). \end{aligned}$$

Definition 1.2.2 *Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a function. T is a* linear transformation *of $\mathbb{R}^2$ if it respects both addition and scalar multiplication, i.e. if*

$$\begin{aligned} T(u+v) &= T(u) + T(v) \text{ for all } u, v \in \mathbb{R}^2, \text{ and} \\ T(ku) &= kT(u) \text{ for all } u \in \mathbb{R}^2 \text{ and } k \in \mathbb{R}. \end{aligned}$$

Note: Suppose $u = (a,b) \in \mathbb{R}^2$. Then $a, b \in R$ and $u = (a,0) + (0,b)$. We have $(a,0) = a(1,0)$ and $(0,b) = b(0,1)$. So

$$u = (a,b) = a(1,0) + b(0,1).$$

Thus any element of $\mathbb{R}^2$ can be written as the sum of a scalar multiple of $(1,0)$ and a scalar multiple of $(0,1)$.
The set $\{(1,0),(0,1)\}$ is called the *standard basis* of $\mathbb{R}^2$.

Claim: If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation and we know $T(1,0)$ and $T(0,1)$, we can write down $T(x,y)$ for any $(x,y) \in \mathbb{R}^2$.

Example 1.2.3 *Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation. If $T(1,0) = (-1,2)$ and $T(0,1) = (3,4)$, what is*

$$\text{(i) } T(-1,5)? \qquad \text{(ii) } T(3,-1/2)\ ?$$

Solution : (i)

$$\begin{aligned} T(-1,5) &= T((-1,0) + (0,5)) \\ &= T(-1(1,0) + 5(0,1)) \\ &= T(-1(1,0)) + T(5(0,1)) \\ &= -1T(1,0) + 5T(0,1) \\ &= -1(-1,2) + 5(3,4) \\ &= (1,-2) + (15,20) \\ &= (16,18). \end{aligned}$$

(ii) Answer is $(-9/2, 4)$ - Exercise.

In general we have the following statement.

Theorem 1.2.4 *Suppose that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation satisfying $T(1,0) = (a,b)$ and $T(0,1) = (c,d)$. Then if (x,y) is any element of $\mathbb{R}^2$, we have*

$$T(x,y) = (ax + cy, bx + dy).$$

Proof:

$$\begin{aligned} T(x,y) &= T((x,0) + (0,y)) \\ &= T(x(1,0) + y(0,1)) \\ &= T(x(1,0)) + T(y(0,1)) \\ &= xT(1,0) + yT(0,1) \\ &= x(a,b) + y(c,d) \\ &= (ax,bx) + (cy,dy) \\ &= (ax+cy, bx+dy). \end{aligned}$$