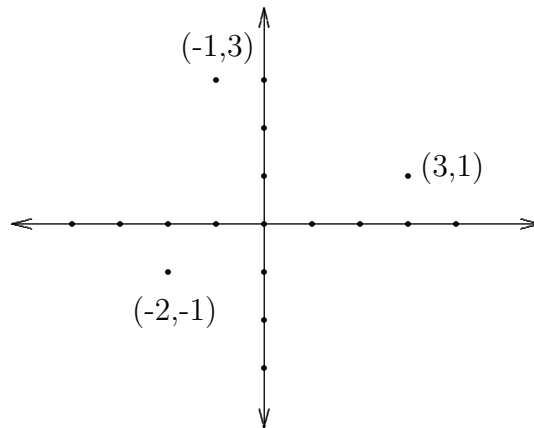


Chapter 1

Matrices and Linear Transformations

1.1 The Euclidean Plane \mathbb{R}^2

The 2-dimensional plane is described by a pair of axes labelled X and Y . Each point in the plane corresponds to the *ordered pair* of real numbers, consisting firstly of its X -coordinate and secondly of its Y -coordinate.



Note: The set of real numbers is denoted by \mathbb{R} and can be considered just as a set or as the real number line.

The set of *ordered pairs* of real numbers is denoted by \mathbb{R}^2 :

$$\mathbb{R}^2 = \{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}\}.$$

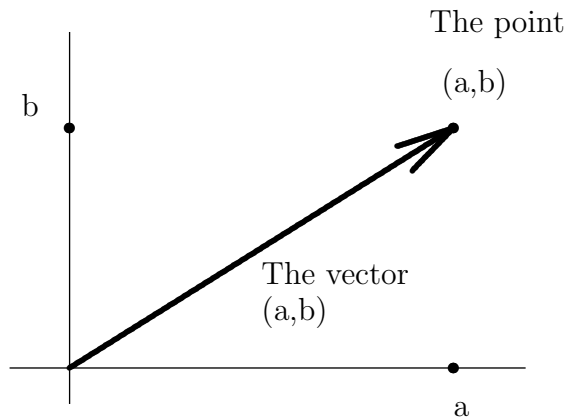
We can think of \mathbb{R}^2 either as this set (abstract, set-theoretic description) or as the set of points in the Cartesian plane above (more concrete, geometric description).

Question: What do we understand by the object (a, b) (e.g. $(1, -3), (3/5, 100), (-\pi, \sqrt{3})$ etc.)?

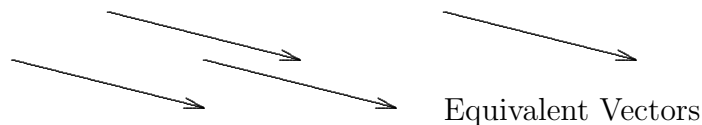
It will be useful to understand this object in (at least) three different ways :

1. Just as an ordered pair of numbers.
2. As the *point* in the plane with X -coordinate a and Y -coordinate b .

3. As the *vector* \vec{v} in the plane directed from the origin $O(0, 0)$ to the point (a, b) . Vectors in \mathbb{R}^2 are line segments with a direction. If $\vec{v} = (a, b)$ is considered as a vector, the numbers a and b are referred to as the X -component and Y -component of \vec{v} .



Note on Vectors: The vector $\vec{v} = (a, b)$ is the line segment directed from $(0, 0)$ to (a, b) . Any directed line segment in which the terminal point (end point) can be reached from the initial point (start point) by travelling a units along the X -axis (right or left according as a is positive or negative) and b units along the Y -axis (up if b is positive, down if negative) is said to be *equivalent* to \vec{v} . Equivalent vectors are considered to be the same. This means : given a vector \vec{v} , we can move it around in \mathbb{R}^2 as long as we do not change its length or direction.



ADDITION IN \mathbb{R}^2

Let (a_1, b_1) and (a_2, b_2) be elements of \mathbb{R}^2 . We define their *sum* in \mathbb{R}^2 by

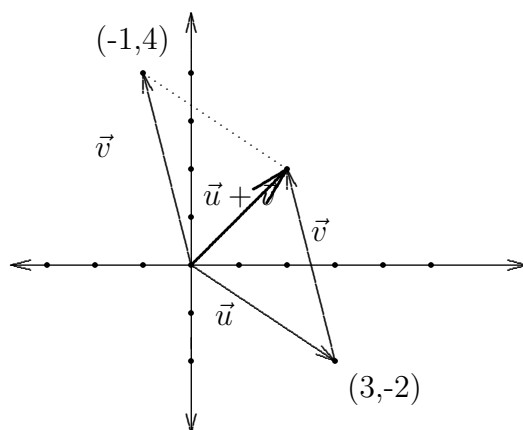
$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2).$$

Examples $(-3, 5) + (2, 4) = (-3 + 2, 5 + 4) = (-1, 9)$.

$$(\sqrt{2}, 1) + (-\sqrt{3}, \sqrt{5}) = (\sqrt{2} - \sqrt{3}, 1 + \sqrt{5}).$$

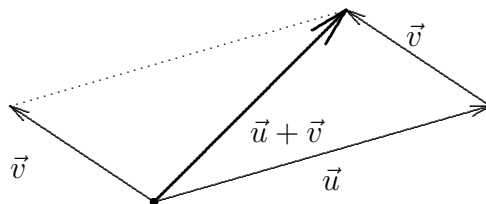
Geometric Interpretation

Points: The point with coordinates $(a_1 + a_2, b_1 + b_2)$ is the fourth vertex of the parallelogram that has (a_1, b_1) , $(0, 0)$ and (b_1, b_2) as three consecutive vertices around its perimeter.



If \vec{v}_1 and \vec{v}_2 are interpreted as vectors originating at the origin, their sum $\vec{v}_1 + \vec{v}_2$ is a diagonal of the parallelogram having \vec{v}_1 and \vec{v}_2 as two sides.

Alternatively : if \vec{v}_1 and \vec{v}_2 are vectors, their sum $\vec{v}_1 + \vec{v}_2$ can be defined as follows - position \vec{v}_2 with its initial point at the terminal point of \vec{v}_1 . The arrow directed from the initial point of \vec{v}_1 to the terminal point of \vec{v}_2 then is the vector $\vec{v}_1 + \vec{v}_2$.



SCALAR MULTIPLICATION

In the context of vectors in \mathbb{R}^2 , real numbers are often referred to as *scalars*. We can multiply an element of \mathbb{R}^2 by a scalar as follows : let $v \in \mathbb{R}^2, v = (a, b)$. If $k \in \mathbb{R}$ then $kv \in \mathbb{R}^2$ is defined by

$$kv = k(a, b) = (ka, kb).$$

Geometric Interpretation: Consider \vec{v} to be a vector with initial point at $O(0, 0)$. Then kv is a vector whose length is $|k| \times (\text{length of } v)$, and whose direction is

- the same as that of v if $k > 0$
- opposite to that of v if $k < 0$.

Note: For a real number k , $|k|$ denotes the absolute value of k . This is equal to k if $k \geq 0$, and equal to $-k$ (a positive number) if k is negative. For example the absolute value of -2 is 2; $|-2| = 2$. The absolute value of a non-zero real number is always positive.

Once equipped with these operations of addition and scalar multiplication, \mathbb{R}^2 is no longer just a set - it has *algebraic structure*. (In fact \mathbb{R}^2 is an example of a *vector space*).