Section 3.1: Introduction to sequences and series

Question 51

Does it make sense to talk about the "number"

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots ?$$

- $1 + \frac{1}{4} = 1.25$
- $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.423611$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10)^2} \approx 1.549767$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(200)^2} \approx 1.639947$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10000)^2} \approx 1.644834$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(100000)^2} \approx 1.644924$

$$\frac{\pi^2}{6} \approx 1.644934$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges to the number $\frac{\pi^2}{6}$ (we will have precise definitions for the highlighted terms a bit later).

This fact is remarkable - there is no obvious connection between π and squares of the form $\frac{1}{n^2}$; moreover all the terms in the series are rational but $\frac{\pi^2}{6}$ is certainly not.
This example gives us in principle a way of calculating the digits of π or at least of π^2. (In practice there are similar but better ways, as the convergence in this example is very slow).

Another Example

Example 52

What about

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots ?$$

Try experimenting with initial segments again :

- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50} \approx 4.4992$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100} \approx 5.1874$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{1000} \approx 7.4855$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50000} \approx 11.3970$

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

Another Example . . .

Example 53

What about

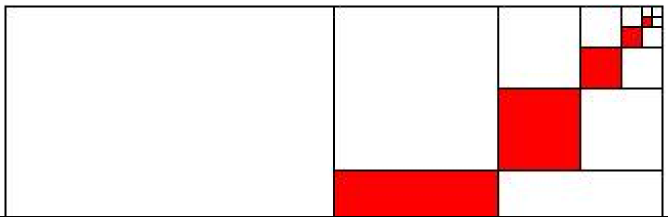
$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots ?$$

Experimenting reveals

- $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$
- $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} = \frac{341}{1024} \approx 0.33301$
- $\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots + \frac{1}{2^{14}} \approx 0.3333$

These calculations can be verified directly using properties of sums of geometric progressions. It appears that this series is converging (quite fast) to $\frac{1}{3}$.

The following picture gives some graphical evidence for this hypothesis.



A last example

Example 54

Does it make sense to talk about

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

as a function of x?

If it does, then f must have a domain (consisting of some or all of the real numbers?) and substituting these values in to the definition in place of x must somehow make sense.

- $x = 0$: $f(0) = 0$
- $x = \frac{\pi}{2}$: $f(\frac{\pi}{2}) \approx 0.9999$ (six terms)
- $x = \frac{\pi}{6}$: $f(\frac{\pi}{6}) \approx 0.5000$ (six terms)
- $x = \frac{\pi}{3}$: $f(\frac{\pi}{3}) \approx 0.8660$ (six terms) ($\frac{\sqrt{3}}{2} \approx 0.8660$)

In all cases we get (just from the first six terms) something very close to $\sin x$.