Section 2.5 : The Completeness Axiom in $\mathbb{R}$

The rational numbers and real numbers are closely related.

- The set $\mathbb{Q}$ of rational numbers is countable and the set $\mathbb{R}$ of real numbers is not, and in this sense there are many more real numbers than rational numbers.
- However, $\mathbb{Q}$ is "dense" in $\mathbb{R}$. This means that every interval of the real number line, no matter how short, contains infinitely many rational numbers. This statement has a practical impact as well, which we use all the time.

Lemma 45

Every real number (whether rational or not) can be approximated by a rational number with a level of accuracy as high as we like.

Justification for this claim

- 3 is a rational approximation for π.
- 3.1 is a closer one.
- 3.14 is closer again.
- 3.14159 is closer still.
- 3.1415926535 is even closer than that,

and we can keep improving on this by truncating the decimal expansion of π at later and later stages.
If we want a rational approximation that differs from the true value of π by less than $\boxed{10^{-20}}$ we can truncate the decimal approximation of π at the 21st digit after the decimal point. This is what is meant by "a level of accuracy as high as we like" in the statement of the lemma.

Notes

1. The fact that all real numbers can be approximated with arbitrary closeness by rational numbers is used all the time in everyday life. Computers basically don't deal with all the real numbers or even with all the rational numbers, but with some specified level of precision. They really work with a subset of the rational numbers.
2. The sequence

$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ 3.141592,\ \ldots$$

   is a list of numbers that are steadily approaching π. The terms in this sequence are increasing and they are approaching π. We say that this sequence converges to π and we will investigate the concept of convergent sequences in Chapter 3.
3. We haven't looked yet at the question of how the numbers in the above sequence can be calculated, i.e. how we can get our hands on better and better approximations to the value of the irrational number π. That's another thing that we will look at in Chapter 3.

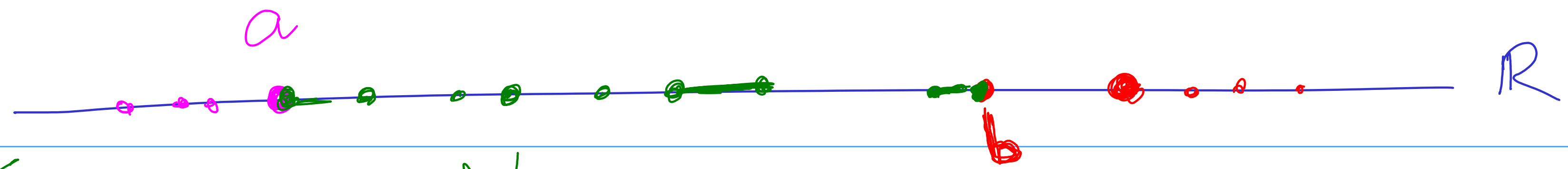
Upper and Lower Bounds

The goal of this last section of Chapter 2 is to pinpoint one essential property of subsets of $\mathbb{R}$ that is not shared by subsets of $\mathbb{Z}$ or of $\mathbb{Q}$. We need a few definitions and some terminology in order to describe this.

Definition 46

Let S be a subset of $\mathbb{R}$. An element b of $\mathbb{R}$ is an upper bound for S if $x \leq b$ for all $x \in S$. An element a of $\mathbb{R}$ is a lower bound for S if $a \leq x$ for all $x \in S$.

So an upper bound for S is a number that is to the right of all elements of S on the real line, and a lower bound for S is a number that is to the left of all points of S on the real line. Note that if b is an upper bound for S, then so is every number b' with $b < b'$. If a is a lower bound for S then so is every number a' with $a' < a$. So if S has an upper bound at all it has infinitely many upper bounds, and if S has a lower bound at all it has infinitely many lower bounds.



S: green points

b is an upper bound for S if no point of S is to the right of b on the number line.

a is a lower bound for S if no element of S is to the left of a.

Upper and Lower Bounds

Definition 47

Let S be a subset of $\mathbb{R}$. An element b of $\mathbb{R}$ is an upper bound for S if $x \leq b$ for all $x \in S$. An element a of $\mathbb{R}$ is a lower bound for S if $a \leq x$ for all $x \in S$.

Recall that

- S is bounded above if it has an upper bound,
- S is bounded below if it has a lower bound,
- S is bounded if it is bounded both above and below.

In this section we are mostly interested in sets that are bounded on at least one side.

Maximum and minimum elements

$\max(\{1,3,24\}) = 24$ $\max([0,1]) = 1$ $\mathbb{Q}$ has no maximum

$(0,1)$ has no maximum

Definition 48

Let S be a subset of $\mathbb{R}$. If there is a number m that is both an element of S *and* an upper bound for S, then m is called the maximum element of S and denoted $\max(S)$. The maximum element of S is the greatest element of S.

If there is a number l that is both an element of S and a lower bound for S, then l is called the minimum element of S and denoted by $\min(S)$.

Notes

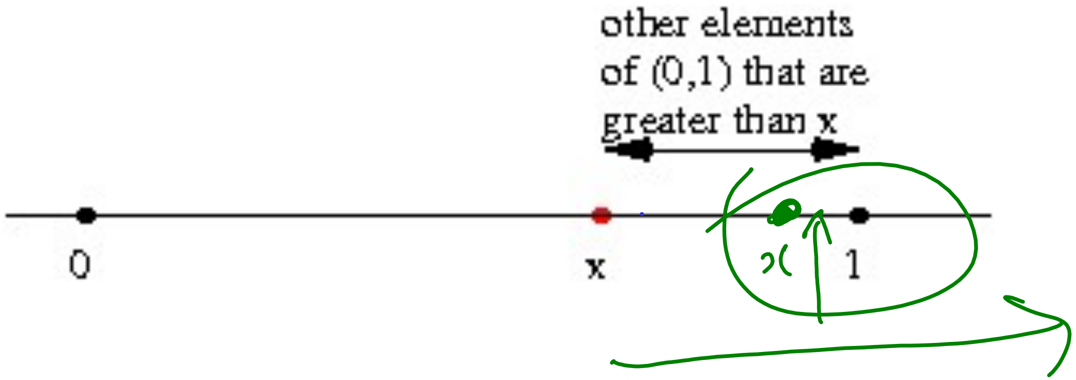
A set can have at most one maximum (or minimum) element.

Pictorially, on the number line, the maximum element of S is the rightmost point that belongs to S, if such a point exists. The minimum element of S is the leftmost point on the number line that belongs to S, if such a point exists.

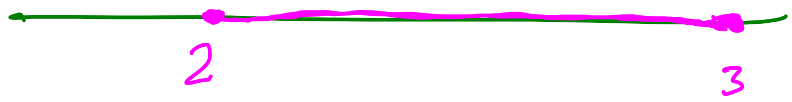
Not every set has a maximum element

There are basically two reasons why a subset S of $\mathbb{R}$ might fail to have a maximum element. First, S might not be bounded above - then it certainly won't have a maximum element.

Secondly, S might be bounded above, but might not contain an element that is an upper bound for itself. Take for example an open interval like $(0,1)$. This set is certainly bounded above. However, take any element x of $(0,1)$. Then x is a real number that is strictly greater than 0 and strictly less than 1. Between s and 1 there are more real numbers all of which belong to $(0,1)$ and are greater than x. So x cannot be an upper bound for the interval $(0,1)$.



Maximum and Minimum Elements



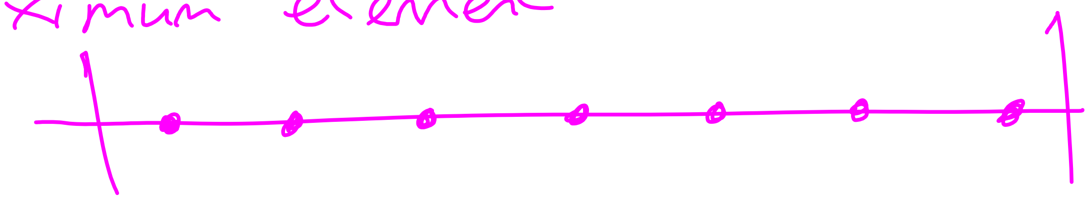
An open interval like $(0, 1)$, although it is bounded, has no maximum element and no minimum element.
An example of a subset of $\mathbb{R}$ that *does* have a maximum and a minimum element is a *closed* interval like $[2, 3]$. The minimum element of $[2, 3]$ is 2 and the maximum element is 3.

non-empty

Remark : Every finite subset of $\mathbb{R}$ has a maximum element and a minimum element.

non-empty

Remark Every bounded subset of $\mathbb{Z}$ has a minimum and a maximum element



Supremum and Infimum

(0,1)

For bounded subsets of $\mathbb{R}$, there are notions called the supremum and infimum that are closely related to maximum and minimum. Every subset of $\mathbb{R}$ that is bounded above has a supremum and every subset of $\mathbb{R}$ that is bounded below has an infimum.

S upper bounds for S

Definition 49 (The Axiom of Completeness for $\mathbb{R}$)

Let S be a subset of $\mathbb{R}$ that is bounded above. Then the set of all upper bounds for S has a minimum element. This number is called the supremum of S and denoted $\sup(S)$.

Let S be a subset of $\mathbb{R}$ that is bounded below. Then the set of all lower bounds for S has a maximum element. This number is called the infimum of S and denoted $\inf(S)$.

Notes

1. The supremum of S is also called the least upper bound (lub) of S.
2. The infimum of S is also called the greatest lower bound (glb) of S.

The definition above is simultaneously a definition of the terms supremum and infimum and a statement of the Axiom of Completeness for the real numbers.

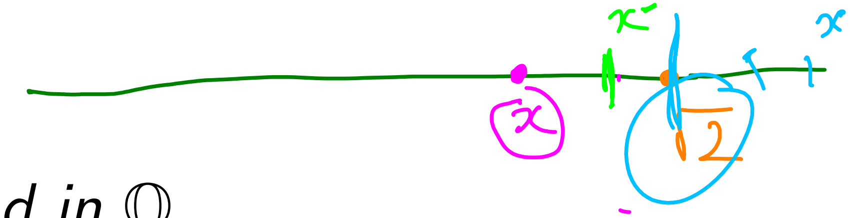
To see why this statement says something special about the real numbers, temporarily imagine that the only number system available to us is $\mathbb{Q}$, the set of rational numbers. Look at the set

$$\boxed{S := \{x \in \mathbb{Q} : x^2 < 2\}.}$$

If we take a set of rational numbers that's bounded above, it need not have a least upper bound in $\mathbb{Q}$

$S := \{x \in \mathbb{Q} : x^2 < 2\}$.

So S consists of all those rational numbers whose square is less than 2. It is bounded above, for example by 2.
The positive elements of S are all those positive rational numbers that are less than the real number $\sqrt{2}$.



Claim: *S does not have a least upper bound in $\mathbb{Q}$.*
To see this, suppose that x is a rational number that is a candidate for being the least upper bound of S in $\mathbb{R}$. $x^2 \neq 2$ since x is rational

- If $x^2 < 2$, then there is a gap in the number line between x and $\sqrt{2}$, and in this gap are rational numbers that are greater than x but still less than $\sqrt{2}$. So x is not an upper bound of S.
- If $x^2 > 2$, then there is a gap in the number line between $\sqrt{2}$ and x, and in this gap are rational numbers that are still upper bounds of S but are less than x.

$S := \{x \in \mathbb{Q} : x^2 < 2\}$

If we consider the same set S as a subset of $\mathbb{R}$, we can see that $\sqrt{2}$ is the supremum of S in $\mathbb{R}$ (and $-\sqrt{2}$) is the infimum of S in $\mathbb{R}$.

This example demonstrates that the Axiom of Completeness does not hold for $\mathbb{Q}$, i.e. a bounded subset of $\mathbb{Q}$ need not have a supremum in $\mathbb{Q}$ or an infimum in $\mathbb{Q}$.

A question from the 2014 exam . . .

Question 50

Let $S = \left\{ \frac{2n+4}{3n} : n \in \mathbb{Z},\ n \geq 1 \right\}$.

1. *List four elements of* S.
2. *Identify, with explanation, the maximum element of* S.
3. *Show that* S *has no minimum element, and determine the infimum of* S.

Learning Outcomes for Section 2.5

After studying this section you should be able to

- State what it means for a subset of $\mathbb{R}$ to be *bounded* (or *bounded above* or *bounded below*).
- Define the terms maximum, minimum, supremum and infimum and explain the connections and differences between them.
- State the Axiom of Completeness.
- Determine whether a set presented like the one in the problem above is bounded (above and/or below) or not and identify its maximum/minimum/infimum/supremum as appropriate, with explanation.