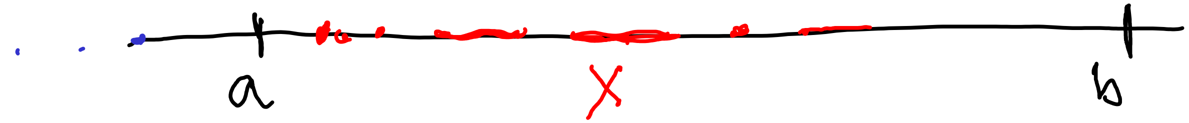
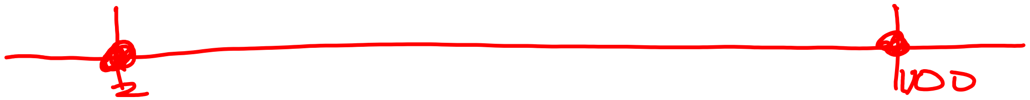
Bounded and unbounded subsets of $\mathbb{R}$



Basically, a subset X of $\mathbb{R}$ is bounded if, on the number line, its elements do not extend indefinitely to the left or right. In other words there exist real numbers a and b with $a < b$, for which all the points of X are in the interval (a, b).

Definition

Let X be a subset of $\mathbb{R}$. Then X is bounded below if there exists a real number a with $a \leq x$ for all elements x of X. (Note that a need not belong to X here).
The set X is bounded above if there exists a real number b with $x < b$ for elements x of X. (Note that b need not belong to X here).
The set X is bounded if it is bounded above and bounded below (otherwise it's unbounded).



Example

Bounded neither above nor below
No matter how far we travel to the right on the number line, we will always encounter more elements of $\mathbb{Q}$.

1. $\mathbb{Q}$ *is unbounded.*
2. $\mathbb{N}$ *is bounded below but not above.*
3. $(0,1)$, $[0,1]$, $[2,100]$ *are bounded.*
4. $\{\cos x : x \in \mathbb{R}\}$ *is bounded, since* $\cos x$ *can only have values between* -1 *and* 1*.*
5. *All finite subsets of* $\mathbb{R}$ *are bounded, and some infinite subsets are.*

$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$
0 is a lower bound for $\mathbb{N}$

Question: Is it possible for a bounded set to have the same cardinality as an unbounded set?

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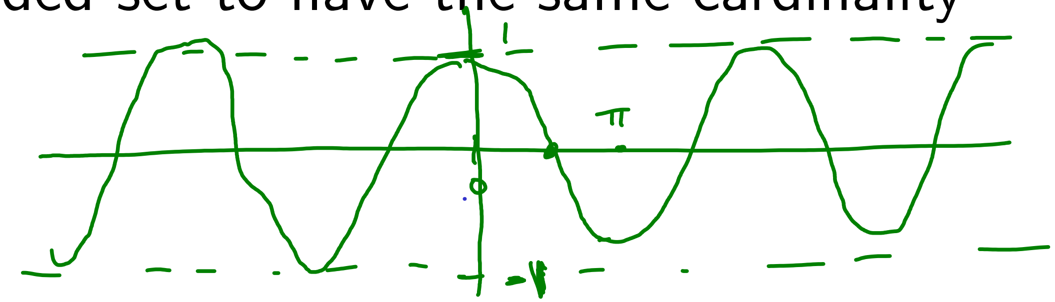


Open intervals

The open interval from a to b (for real numbers a and b with $a < b$) is $(a,b) = \{x : a < x < b\}$ (a & b are not included)

In our next example we show that the set of all the real numbers has the same cardinality as an open interval on the real line.
First we note that all open intervals have the same cardinality as each other. $(10,100)$

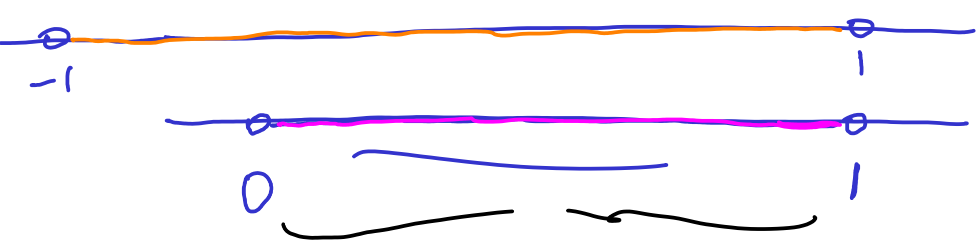
Exercise 43

Show that the the open interval $\boxed{(0,1)}$ *has the same cardinality as*

* 1. *The open interval* $\boxed{(-1,1)}$
2. *The open interval* $(1,2)$
3. *The open interval* $(2,6)$.

$x \longrightarrow 2x$ gives us a bijective correspondence between $(0,1)$ and $(0,2)$. Then subtract 1 to set from $(0,2)$ to $(-1,1)$

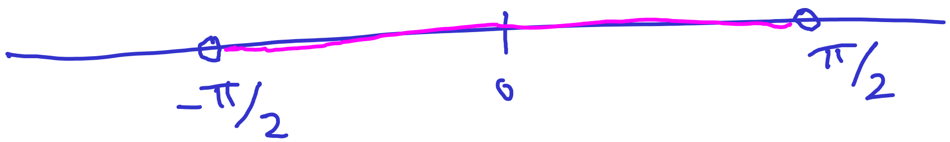
$x \longrightarrow \boxed{2x-1}$ is a bijection between $(0,1)$ and $(-1,1)$



$(-1,1)$ - open interval of length 2

$\boxed{(0,1)}$ - open interval of length 1

The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Example

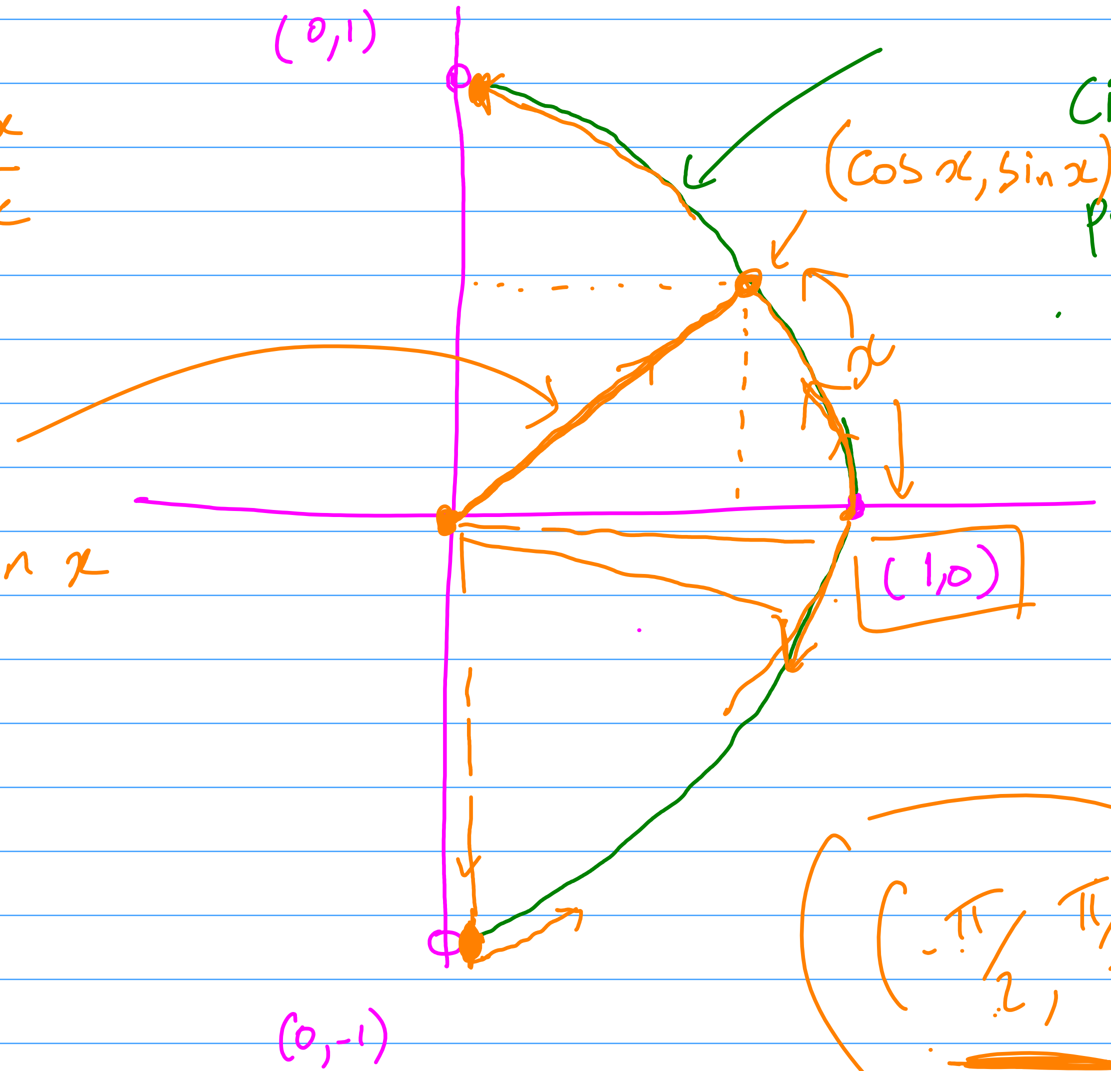
Show that $\mathbb{R}$ *has the same cardinality as the open interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

In order to do this we have to establish a bijective correspondence between the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the full set of real numbers. An example of a function that provides us with such a bijective correspondence is familiar from calculus/trigonometry.

$$\tan x = \frac{\sin x}{\cos x}$$

Part of the unit circle whose points have positive x-coordinates

Slope of this line segment is $\tan x$



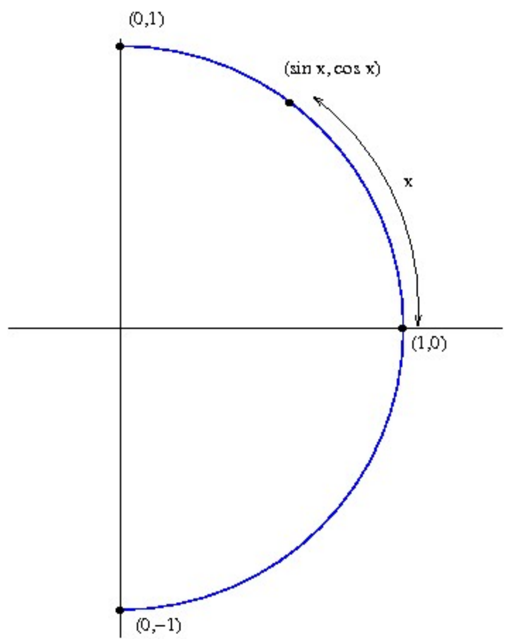
$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longleftrightarrow \mathbb{R}$$

$(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$

Recall that for a number x in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\tan x$ is defined as follows: travel from $(1, 0)$ a distance $|x|$ along the circumference of the unit circle, anti-clockwise if x is positive and clockwise if x is negative. We arrive at a point which is in the right-hand side of the unit circle.

Now $\tan x$ is the slope of the line that connects the origin to this point (whose y and x-coordinates are $\sin x$ and $\cos x$ respectively).



$\tan x$ gives a bijection

Now $\tan 0 = 0$, and as x increases from 0 towards $\frac{\pi}{2}$, the line segment in question rotates about the origin into the first quadrant, its slope increases continuously from zero, without limit as x approaches $\frac{\pi}{2}$. So every positive real number is the tan of exactly one x in the range $(0, \frac{\pi}{2})$.

For the same reason, the values of $\tan x$ include every negative real number exactly once as x runs between 0 and $-\frac{\pi}{2}$.

Thus for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the correspondence

$$x \longleftrightarrow \tan x$$

establishes a bijection between the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the full set of real numbers.

We conclude that the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has the same cardinality as $\mathbb{R}$.

Note: This assertion is unrelated to the concept of countability discussed earlier.

Some Remarks

1. We don't know yet if $\mathbb{R}$ (or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) has the same cardinality as $\mathbb{N}$ - we don't know if $\mathbb{R}$ is countable.
2. The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ may seem like an odd choice for an example like this. However, note that the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is in bijective correspondence with the interval $(-1, 1)$, via the function that just multiplies everything by $\frac{2}{\pi}$.

Learning outcomes for Section 2.3

This section contains some very challenging concepts. You will probably need to invest some serious intellectual effort in order to arrive at a good understanding of them. This is an effort worth making as it has the potential to really expand your view of what mathematics is about.
After studying this section you should be able to

- Discuss the concept of bijective correspondence for infinite sets;
- Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality by exhibiting a bijective correspondence between them;
- Explain what is meant by a *countable* set and show that $\mathbb{Q}$ is countable;
- Exhibit a bijective correspondence between $\mathbb{R}$ and the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and hence show that $\mathbb{R}$ has the same cardinality as the interval (a, b) for any real numbers a and b with $a < b$.