The real number line

Now imagine an infinite straight line, on which the integers are marked (in order) by an infinite set of evenly spaced dots.

Imagine that the rational numbers have also been marked by dots, so that the dot representing $\frac{3}{2}$ is halfway between the dot representing 1 and the dot representing 2, and so on.

At this stage a lot of dots have been marked - every stretch of the line, no matter how short, contains an infinite number of marked dots. However, many points on the line remain unmarked. For example, somewhere between the dot representing the rational number 1.4142 and the dot representing the rational number 1.4143 is an unmarked point that represents the real number $\sqrt{2}$.

The set $\mathbb{R}$ of *real numbers* is the set of numbers corresponding to *all* points on the line, marked or not.

Not all real numbers are rational

Example

$\sqrt{3}$ *is irrational.*

Proof (by contradiction). Suppose that $\sqrt{3}$ is rational and write

$$\sqrt{3} = \frac{m}{n},$$

where m and n are positive integers with no common integer factors. Then

$$3 = \frac{m^2}{n^2} \implies m^2 = 3n^2.$$

This means that m^2 is a multiple of 3, and so m is a multiple of 3, which means that m^2 is actually a multiple of 9; write $m^2 = 9k$, for some $k \in \mathbb{Z}$. Then

$$m^2 = 9k = 3n^2 \implies n^2 = 3k,$$

so n^2 is a multiple of 3, hence so is n. But now both m and n are multiples of 3, which means there is no way to write $\sqrt{3}$ in the form $\frac{m}{n}$ for integers m and n with no common factors.

Rational and Irrational Numbers

Remark: Because the examples of irrational numbers that are usually cited are things like $\sqrt{2}, \pi$ and e, you could get the impression that irrational numbers are special and rare.

This is far from being true!

In a very precise way that we will see later, the irrational numbers are more numerous that the rational numbers.

Exercise 39

Write down five irrational numbers between 4 and 5.

Exercise about rational and irrational numbers

Exercise 40

Suppose that a is a rational number and b is an irrational number.

- *Might $a+b$ be irrational?*
- *Must $a+b$ be irrational?*
- *Might ab be irrational?*
- *Must ab be irrational?*
- *Might the product of two rational numbers be rational?*
- *Must the product of two rational numbers be rational?*
- *Might the product of two irrational numbers be irrational?*
- *Must the product of two irrational numbers be irrational?*

Views of $\mathbb{R}$

To conclude this section we propose two different ways of thinking about the set of real numbers.

Arithmetic description: The set $\mathbb{R}$ of real numbers consists of *all* numbers that can be written as (possibly non-terminating and possibly non-repeating) decimals.

This description is conceptually valuable but not of much practical use.

All numbers that can be expressed as decimals means *all* numbers that can be written as sequences of the digits $0,1,\ldots,9$ (with a decimal point somewhere) with no pattern of repetition necessary in the digits. In the universe of all such things, the ones that terminate or have a repeating pattern from some point onwards are special and rare. These are the rational numbers. The ones that have all zeroes after the decimal point are even more special - these are the integers.

Views of $\mathbb{R}$

To conclude this section we propose two different ways of thinking about the set of real numbers.

Geometric description The set $\mathbb{R}$ of real numbers is the set of *all* points on the number line. This is a *continuum* - there are no gaps in the real numbers and no point on the line that doesn't correspond to a real number.

Note: As this course proceeds you will need to know what the symbols $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ mean and be able to recall this information easily. You'll need to be familiar with all the notation involving sets etc. that is used in this section and to be able to use it in an accurate way.

Learning outcomes for Section 2.1

After studying this section you should be able to:

- Use the notation $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ correctly and reliably, and describe the elements of these sets.
- Explain how these sets appear on the number line, and point out some important differences between them.

Section 2.2: Subsets of $\mathbb{R}$

We consider the notions of finite and infinite sets, and the cardinality of a set.

Reasonable goals for this section are to become familiar with these ideas and to practice interpreting descriptions of sets that are presented in terse mathematical notation (this means, amongst other things, distinguishing between different kinds of brackets: $\{\}$, $[\,]$, $(\,)$, etc.).

Finite and Infinite Sets

Definition

A set is finite if it is possible to list its distinct elements one by one, and this list comes to an end.
A set is infinite if any attempt at listing its distinct elements continues indefinitely.

Example

$\{1, 2, 3, 4, 5\}$ is a *finite* set - its only elements are the integers 1,2,3,4,5, there are five of them.

Finite and Infinite Sets

Definition

A set is finite if it is possible to list its distinct elements one by one, and this list comes to an end.
A set is infinite if any attempt at listing its distinct elements continues indefinitely.

Example

The interval $[1, 3]$ is an *infinite* set - it consists of *all* the real numbers that are at least equal to 1 and at most equal to 3.

$$[1, 3] := \{x \in \mathbb{R} : 1 \leq x \leq 3\}.$$

Note: The symbol ":=" here means this is a statement of the *definition* of $[1, 3]$.

Finite and Infinite Sets

Definition

A set is finite if it is possible to list its distinct elements one by one, and this list comes to an end.
A set is infinite if any attempt at listing its distinct elements continues indefinitely.

Example

$\mathbb{Z}$ and $\mathbb{Q}$ are *infinite* sets.

Finite and Infinite Sets

Definition

A set is finite if it is possible to list its distinct elements one by one, and this list comes to an end.
A set is infinite if any attempt at listing its distinct elements continues indefinitely.

Example

The set of real solutions of the equation

$$x^5 + 2x^4 - x^2 + x + 17 = 0$$

is a *finite* set. We don't know how many elements it has, but it has at most five, since each one corresponds to a factor of degree 1 of this polynomial of degree 5.

Finite and Infinite Sets

Definition

A set is finite if it is possible to list its distinct elements one by one, and this list comes to an end.
A set is infinite if any attempt at listing its distinct elements continues indefinitely.

Example

The set of prime numbers is infinite.
A pair of twin primes is a pair of primes that differ by 2: e.g. 3 and 5, 11 and 13, 59 and 61. It is not known whether the set of pairs of twin primes is finite or infinite.

Cardinality

Definition

The cardinality of a finite set S, denoted $|S|$, is the number of elements in S.

Example

1. If $S = \{5, 7, 8\}$ then $|S| = 3$.
2. $|\{4, 10, \pi\}| = 3$
3. $|\{x \in \mathbb{Z} : \pi < x < 3\pi\}| = 6$.
   Note: $\{x \in \mathbb{Z} : \pi < x < 3\pi\} = \{4, 5, 6, 7, 8, 9\}$.
4. The cardinality of $\mathbb{Q}$ is infinite.

Remarks

1. The notation "$|\cdot|$" is severely overused in mathematics. If x is a real number, $|x|$ means the absolute value of x. If S is a set, $|S|$ means the cardinality of S. If A is a matrix $|A|$ means the determinant of A. It is supposed to be clear from the context what is meant.
2. Defining the concept of cardinality for infinite sets is trickier, since you can't say how many elements they have. We will be able to say though what it means for two infinite sets to have the same (or different) cardinalities.

A silly example

Example

In a hotel, keys for all the guest rooms are kept on hooks behind the reception desk. If a room is occupied, the key is missing from its hook because the guests have it. If the receptionist wants to know how many rooms are occupied, s/he doesn't have to visit all the rooms to check - s/he can just count the number of hooks whose keys are missing.

In this example, the occupied rooms are in one-to-one correspondence with the empty hooks. This means that each occupied room corresponds to one and only one empty hook, and each empty hook corresponds to one and only one occupied room. So the number of empty hooks is the same as the number of occupied rooms and we can count one by counting the other.

Bijections and bijective correspondence

Definition

Suppose that A and B are sets. Then a one-to-one correspondence or a bijective correspondence between A and B is a pairing of each element of A with an element of B, in such a way that every element of B is matched to exactly one element of A.

Definition

Suppose that A and B are sets. A function $f : A \longrightarrow B$ is called a bijection if

- *Whenever a_1 and a_2 are different elements of A, $f(a_1)$ and $f(a_2)$ are different elements of B.*
- *Every element b of B is the image of some element a of A.*

Cardinality and bijective correspondence

If a bijective correspondence exists between two finite sets, they have the same cardinality. Sometimes, in order to determine the cardinality of a set, it is easiest to determine the cardinality of another set with which we know it is in bijective correpsondence.

Example

How many integers between 1 and 1000 are perfect squares?

Solution: The list of perfect squares in our range begins as follows

$$1,\ 4,\ 9,\ 16, \dots$$

One way to proceed would be to keep writing out successive terms of this sequence until we hit one that exceeds 1000, then delete that one and count the terms that we have. This is more work than we are asked to do, since we don't need the list of squares but just the number of them. Alternatively, we could notice that $(31)^2 = 961$ and $(32)^2 = 1024$.
So the numbers $1^2, 2^2, \dots, (31)^2$ are the only squares in the range 1 to 1000 and there are 31 of them.

Another example of bijective correspondence

This last example shows that it could be possible to know that there is a bijective correspondence between two finite sets, without knowing the cardinality of either of them.

Example

Show that the equations

$$x^3 + 2x + 4 = 0 \text{ and } x^3 + 3x^2 + 5x + 7 = 0$$

have the same number of real solutions.

$x^3 + 2x + 4 = 0$ and $x^3 + 3x^2 + 5x + 7 = 0$

Solution: One way of doing this is to demonstrate a bijective correspondence between their sets of real solutions. We can write

$$\begin{aligned} x^3 + 3x^2 + 5x + 7 &= (x^3 + 3x^2 + 3x + 1) + 2x + 6 \\ &= (x+1)^3 + (2x+2) + 4 \\ &= (x+1)^3 + 2(x+1) + 4. \end{aligned}$$

This means that a real number a is a solution of the second equation if and only if

$$(a+1)^3 + 2(a+1) + 4 = 0$$

i.e. if and only if $a+1$ is a solution of the first equation.
The correspondence $a \longleftrightarrow a+1$ is a bijective correspondence between the solution sets of the two equations. So they have the same number of real solutions.
Note: This number is at least 1 and at most 3. Why?

Learning outcomes for Section 2.2

After studying this section you should be able to

- Explain what is meant by the cardinality of a set;
- Read and interpret descriptions of different subsets of $\mathbb{R}$ presented using different standard notations. Decide what the elements of these sets are and whether the sets are finite or infinite;
- Explain what is meant by a *bijective correspondence* and give examples to support your explanation.