Section 1.5 : Improper Integrals

Suppose that $f(x)$ is a continuous function that satisfies

$$\lim_{x \to \infty} f(x) = 0;$$

for example $f(x) = e^{-x}$ has this property. Then we can consider the total area between the graph $y = f(x)$ and the X-axis, to the right of (for example) $x = 1$. This area is denoted by

$$\int_1^{\infty} f(x)\, dx$$

and referred to as an improper integral. For a given function, it is not clear whether the area involved is finite or infinite (if it is infinite, the improper integral is said to diverge or to be divergent). One question that arises is how we can determine if the relevant area is finite or infinite, another is how to calculate it if it is finite.

Definition of $\int_a^\infty f(x)\,dx$

Definition 33

If the function f is continuous on the interval $[a, \infty)$, then the *improper integral* $\displaystyle\int_a^\infty f(x)\,dx$ is defined by

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

provided this limit exists. In this case the improper integral is called convergent (otherwise it's divergent).
Similarly, if f is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^a f(x)\,dx := \lim_{b\to-\infty} \int_b^a f(x)\,dx$$

Convergent or Divergent?

So to calculate an improper integral of the form $\int_1^\infty f(x)\,dx$ (for example), we first calculate

$$\int_1^b f(x)\,dx$$

for a general b. This will typically be an expression involving b. We then take the limit as $b \to \infty$.

Example 34

Show that the improper integral $\int_1^\infty \frac{1}{x}\,dx$ *is divergent.*

Solution:

$$\int_1^b \frac{1}{x}\,dx = \ln x|_1^b = \ln b - \ln 1 = \ln b.$$

Since $\ln b \to \infty$ as $b \to \infty$, $\lim_{b\to\infty} \ln b$ does not exist and the integral diverges.

Another Example

Example 35

Evaluate $\displaystyle\int_{-\infty}^{-2} \frac{1}{x^2}\,dx.$

Solution:

$$\int_{b}^{-2} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_{b}^{-2} = \frac{1}{2} + \frac{1}{b}$$

Then $\lim_{b\to-\infty}\left(\frac{1}{2}+\frac{1}{b}\right) = \frac{1}{2}$, and

$$\int_{-\infty}^{-2} \frac{1}{x^2}\,dx = \frac{1}{2}.$$

Yet another example

Example 36

Determine whether $\int_1^{\infty} e^{-2x}\,dx$ *is convergent or divergent, and evaluate it if it is convergent.*

Solution: $\int_1^b e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_1^b = -e^{-2b} + e^{-2}$. Then

$$\int_1^{\infty} e^{-2x}\,dx = \lim_{b\to\infty}(-e^{-2b} + e^{-2}) = e^{-2}.$$

So the integral is convergent and the enclosed area is $\frac{1}{e^2}$.

Another type of improper integral

If the graph $y = f(x)$ has a vertical asymptote for a value of x in the interval $[c, d]$, these needs to be considered when computing the integral $\int_c^d f(x)\,dx$, since this integral describes the area of a region that is infinite in the vertical direction at the asymptote.

- If the vertical asymptote is at the left endpoint c, then we define

$$\int_c^d f(x)\,dx = \lim_{b \to c^+} \int_b^d f(x)\,dx.$$

- If the vertical asymptote is at the right endpoint d, then we define

$$\int_c^d f(x)\,dx = \lim_{b \to d^-} \int_c^b f(x)\,dx.$$

- If the vertical asymptote is at an interior point m of the interval $[c, d]$, then we define

$$\int_c^d f(x)\,dx = \int_c^m f(x)\,dx + \int_m^c f(x)\,dx,$$

and the two improper integrals involving m are handled as above

Final Example

Example 37

Determine whether the improper integral $\int_{-2}^{4} \frac{1}{x^2}\,dx$ *is convergent or divergent.*

What makes this integral improper is the fact that the graph $y = \frac{1}{x^2}$ has a vertical asymptote at $x = 0$. Thus

$$\int_{-2}^{4} \frac{1}{x^2}\,dx = \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{4} \frac{1}{x^2}\,dx$$

Final Example (continued)

For the first of these two integrals we have

$$\int_{-2}^{0} \frac{1}{x^2}\,dx = \lim_{b\to 0^-} \int_{-2}^{b} \frac{1}{x^2}\,dx$$
$$= \lim_{b\to 0^-} \left(-\frac{1}{x}\right)\Big|_{-2}^{b}$$
$$= \lim_{b\to 0^-} \left(-\frac{1}{b} + \frac{1}{2}\right)$$

Since $\lim_{b\to 0^-}(-\frac{1}{b})$ does not exist, the improper integral $\int_{-2}^{0} \frac{1}{x^2}\,dx$ diverges. This means that the area enclosed between the graph $y = \frac{1}{x^2}$ and the x-axis over the interval $[-2, 0]$ is infinite.

Now that we know that the first of the two improper integrals in our problem diverges, we don't need to bother with the second. The improper integral $\int_{-2}^{4} \frac{1}{x^2}\,dx$ is divergent.

Note of Caution

In this last example, if we had not noticed the vertical asymptote at $x = 0$, we might have proceeded as follows:

$$\int_{-2}^{4} \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)\Big|_{-2}^{4}$$
$$= -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4}$$

This would be wrong! We should check that an integral is not improper before evaluating it in the manner above.

That’s the end of Chapter 1

The first exam question (and first two homework assignments) are based on Chapter 1. The purpose of these items is to assess how well you have achieved the learning outcomes for Chapter 1. Section 1.6 of the lecture notes has some exam advice for Chapter 1, including sample “exam-type” questions with worked solutions and answers.

Things to do:

- Make sure you know how to use all the notation involving integrals, and distinguish definite, indefinite and improper integrals.
- Make sure you understand what the Fundamental Theorem of Calculus, so that you can state it and apply it to examples.
- Practise, practise, practise the techniques of integration (for example use a calculus textbook).