MA180/190/186 Lecture 8 Calculus 4/3/21

From yesterday - integration by parts

$$\boxed{\int uv' \, dx = uv - \int u'v \, dx}$$

One more example on this topic : $\boxed{\int e^x \cos x \, dx}$

Try $u = e^x$ $v' = \cos x$

$u' = e^x$ $v = \sin x$

$$* \quad \int \underset{u}{e^x} \underset{v'}{\cos x} \, dx = \underbrace{e^x \sin x}_{uv} - \underbrace{\int e^x \sin x \, dx}_{u'v}$$

Look at $\boxed{\int e^x \sin x \, dx}$

Use integration by parts again:

$u = e^x$ $v' = \sin x$

$u' = e^x$ $v = -\cos x$

$$\int e^x \sin x \, dx = \boxed{\underset{u\ v}{-e^x \cos x} + \int \underset{u'\ v}{e^x \cos x} \, dx}$$

$$* \quad \int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x \, (+C)$$

$$\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$$

We know how to integrate polynomial functions; for example

$$\int 2x^2 + 3x - 4\,dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x + C.$$

We also know that

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad \int \frac{1}{x+a}\,dx = \ln|x+a| + C$$

and that

$$\int \frac{1}{x^n}\,dx = -\frac{1}{n-1}\frac{1}{x^{n-1}} + C,$$

for $n > 1$.

This section is about integrating rational functions; i.e. quotients in which the numerator and denominator are both polynomials.

Adding Symbolic Fractions

Remark: If we were presented with the task of adding the expressions $\dfrac{2}{x+3}$ and $\dfrac{1}{x+4}$, we would take $(x+3)(x+4)$ as a common denominator and write

$$\boxed{\frac{2}{x+3}+\frac{1}{x+4}} = \frac{2(x+4)}{(x+3)(x+4)}+\frac{1(x+3)}{(x+3)(x+4)}$$
$$= \frac{2(x+4)+1(x+3)}{(x+3)(x+4)} = \boxed{\frac{3x+11}{(x+3)(x+4)}}.$$

Question: Suppose we were presented with the expression $\dfrac{3x+11}{(x+3)(x+4)}$ and asked to rewrite it in the form $\dfrac{A}{x+3}+\dfrac{B}{x+4}$, for *numbers* A and B. How would we do it?

Another Question Why would we want to do such a thing?

Write
$$\frac{3x+11}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}.$$
for numbers A and B?

Then
$$\frac{3x+11}{(x+3)(x+4)} = \frac{A(x+4)}{(x+3)(x+4)} + \frac{B(x+3)}{(x+3)(x+4)} = \frac{(A+B)x+4A+3B}{(x+3)(x+4)}.$$

This means $3x+11 = (A+B)x+4A+3B$ for all x, which means
$$\boxed{A+B=3, \text{ and } 4A+3B=11.}$$

$3A+3B=9$
$4A+3B=11$

$A=2$
$B=1$

Thus $B=1$ and $A=2$. So
$$\frac{3x+11}{(x+3)(x+4)} = \boxed{\frac{2}{x+3} + \frac{1}{x+4}}.$$

An Alternative Method - often more efficient!

We want

$$3x + 11 = A(x+4) + B(x+3), \quad *$$

for all real numbers x. If this statement is true for all x, then in particular it is true when $x = -4$. Setting $x = -4$ gives

$$-12 + 11 = A(0) + B(-1) \implies B = 1. \quad \Rightarrow -1 = -B$$

Setting $x = -3$ gives

$$-9 + 11 = A(1) + B(0) \implies A = 2.$$

Thus

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

Integration using partial fractions

Example 30

Determine $\displaystyle\int \frac{3x+11}{(x+3)(x+4)}\,dx.$

Solution : Write

$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = \boxed{\int \frac{2}{x+3}\,dx + \int \frac{1}{x+4}\,dx}$$

Then

$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = \underline{2\ln|x+3|}+\ln|x+4|+C = \underline{\ln(x+3)^2}+\ln|x+4|+C.$$

Example 31

Determine $\int \frac{x^3+3x+2}{x+1}\,dx$. ← rewrite as $p(x) + \frac{R}{x+1}$

↑ polynomial part

In this example the degree of the numerator exceeds the degree of the denominator, so first apply long division to find the quotient and remainder upon dividing x^3+3x+2 by $x+1$.

We find that the quotient is x^2-x+4 and the remainder is -2. Hence

$$\frac{x^3+3x+2}{x+1} = \boxed{x^2-x+4} + \frac{-2}{x+1}. \qquad *$$

Thus

$$\begin{aligned}\int \frac{x^3+3x+2}{x+1}\,dx &= \int x^2-x+4\,dx - 2\int \frac{1}{x+1}\,dx \\ &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x - 2\ln|x+1| + C.\end{aligned}$$

$$x^2 - x + 4$$

$$x+1 \,\overline{)\, x^3 + 3x + 2}$$

$$x^3 + x^2$$

$$-x^2 + 3x + 2$$

$$-x^2 - x$$

$$4x + 2$$

$$4x + 4$$

$$-2$$

$$\frac{x^3 + 3x + 2}{x + 1} = x^2 - x + 4 - \frac{2}{x+1}$$

Example 32

Determine $\displaystyle\int \frac{x+1}{\boxed{(2x+1)^2(x-2)}}\,dx.$

If we try

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A}{2x+1} + \frac{B}{2x+1} + \frac{C}{x-2}$$

Common denominator is $(2x+1)(x-2)$ not $(2x+1)^2(x-2)$

Solution: In this case the denominator has a repeated linear factor $2x+1$. It is necessary to include both $\dfrac{A}{2x+1}$ and $\dfrac{B}{(2x+1)^2}$ in the partial fraction expansion. We have

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-2}.$$

Common denominator $(2x+1)^2(x-2)$

Then

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A(2x+1)(x-2) + B(x-2) + C(2x+1)^2}{(2x+1)^2(x-2)}.$$

and so

$$\boxed{x+1} = \boxed{A(2x+1)(x-2) + B(x-2) + C(2x+1)^2.} \leftarrow$$

A Harder Example

$2+1 = A(0) + B(0) + C(5)^2$

$$x = 2: \quad 3 = C(5)^2 \qquad C = \frac{3}{25}$$

$-\frac{1}{2}+1$

$$x = -\frac{1}{2}: \quad \frac{1}{2} = B\left(-\frac{5}{2}\right) \qquad B = -\frac{1}{5}$$

$$x = 0: \quad 1 = A(1)(-2) + B(-2) + C(1)^2 \qquad A = -\frac{6}{25}$$

Thus

$$\frac{x+1}{(2x+1)^2(x-2)} = \boxed{\frac{-6/25}{2x+1} + \frac{-1/5}{(2x+1)^2} + \frac{3/25}{x-2}}$$

and

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{6}{25}\int \frac{1}{2x+1}\,dx - \frac{1}{5}\int \frac{1}{(2x+1)^2}\,dx + \frac{3}{25}\int \frac{1}{x-2}\,dx.$$

A Harder Example

Call the three integrals on the right above $I_1,\ I_2,\ I_3$ respectively.

- $I_1: \displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|(+C_1).$
- $I_2: \displaystyle\int \frac{1}{(2x+1)^2}\,dx = -\frac{1}{2(2x+1)}(+C_2).$
- $I_3: \displaystyle\int \frac{1}{x-2}\,dx = \ln|x-2|(+C_3).$

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

Learning outcomes for Section 1.4

At the end of this section you should

- Know the difference between a definite and indefinite integral and be able to explain it accurately and precisely.
- Be able to evaluate a range of definite and indefinite integrals using the following methods:
  - direct methods;
  - suitably chosen substitutions;
  - integration by parts;
  - partial fraction expansions.