Section 1.4.3 : Partial Fraction Expansions

We know how to integrate polynomial functions; for example

$$\int 2x^2 + 3x - 4\,dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x + C.$$

We also know that

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

and that

$$\int \frac{1}{x^n}\,dx = -\frac{1}{n-1}\frac{1}{x^{n-1}} + C,$$

for $n > 1$.

This section is about integrating rational functions; i.e. quotients in which the numerator and denominator are both polynomials.

Adding Symbolic Fractions

Remark: If we were presented with the task of adding the expressions $\frac{2}{x+3}$ and $\frac{1}{x+4}$, we would take $(x+3)(x+4)$ as a common denominator and write

$$\begin{aligned}\frac{2}{x+3}+\frac{1}{x+4} &= \frac{2(x+4)}{(x+3)(x+4)}+\frac{1(x+3)}{(x+3)(x+4)}\\ &= \frac{2(x+4)+1(x+3)}{(x+3)(x+4)}=\frac{3x+11}{(x+3)(x+4)}.\end{aligned}$$

Question: Suppose we were presented with the expression $\frac{3x+11}{(x+3)(x+4)}$ and asked to rewrite it in the form $\frac{A}{x+3}+\frac{B}{x+4}$, for *numbers* A and B. How would we do it?

Another Question Why would we want to do such a thing?

The Partial Fraction Expansion

Write

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}.$$

Then

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A(x+4)}{(x+3)(x+4)} + \frac{B(x+3)}{(x+3)(x+4)} = \frac{(A+B)x+4A+3B}{(x+3)(x+4)}.$$

This means $3x+11 = (A+B)x+4A+3B$ for all x, which means

$$A+B=3, \text{ and } 4A+3B=11.$$

Thus $B=1$ and $A=2$. So

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

An Alternative Method

We want

$$3x + 11 = A(x + 4) + B(x + 3),$$

for all real numbers x. If this statement is true for all x, then in particular it is true when $x = -4$. Setting $x = -4$ gives

$$-12 + 11 = A(0) + B(-1) \implies B = 1.$$

Setting $x = -3$ gives

$$-9 + 11 = A(1) + B(0) \implies A = 2.$$

Thus

$$\frac{3x + 11}{(x + 3)(x + 4)} = \frac{2}{x + 3} + \frac{1}{x + 4}.$$

Integration using partial fractions

Example 30

Determine $\displaystyle\int \frac{3x+11}{(x+3)(x+4)}\,dx.$

Solution : Write

$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = \int \frac{2}{x+3}\,dx + \int \frac{1}{x+4}\,dx$$

Then

$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = 2\ln|x+3| + \ln|x+4| + C = \ln(x+3)^2 + \ln|x+4| + C.$$

Partial fractions with long division

Example 31

Determine $\displaystyle\int \frac{x^3+3x+2}{x+1}\,dx.$

In this example the degree of the numerator exceeds the degree of the denominator, so first apply long division to find the quotient and remainder upon dividing x^3+3x+2 by $x+1$.
We find that the quotient is x^2-x+4 and the remainder is -2. Hence

$$\frac{x^3+3x+2}{x+1} = x^2-x+4+\frac{-2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{x^3+3x+2}{x+1}\,dx &= \int x^2-x+4\,dx - 2\int \frac{1}{x+1}\,dx \\ &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x - 2\ln|x+1| + C.\end{aligned}$$

A Harder Example

Example 32

Determine $\displaystyle\int \frac{x+1}{(2x+1)^2(x-2)}\,dx.$

Solution: In this case the denominator has a repeated linear factor $2x+1$. It is necessary to include both $\dfrac{A}{2x+1}$ and $\dfrac{B}{(2x+1)^2}$ in the partial fraction expansion. We have

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-2}.$$

Then

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A(2x+1)(x-2) + B(x-2) + C(2x+1)^2}{(2x+1)^2(x-2)}.$$

and so

$$x+1 = A(2x+1)(x-2) + B(x-2) + C(2x+1)^2.$$

A Harder Example

$$
\begin{array}{llr}
x = 2: & 3 = C(5)^2 & C = \frac{3}{25} \\
x = -\frac{1}{2}: & \frac{1}{2} = B\left(-\frac{5}{2}\right) & B = -\frac{1}{5} \\
x = 0: & 1 = A(1)(-2) + B(-2) + C(1)^2 & A = -\frac{6}{25}
\end{array}
$$

Thus

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{-6/25}{2x+1} + \frac{-1/5}{(2x+1)^2} + \frac{3/25}{x-2}$$

and

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{6}{25}\int \frac{1}{2x+1}\,dx - \frac{1}{5}\int \frac{1}{(2x+1)^2}\,dx + \frac{3}{25}\int \frac{1}{x-2}\,dx.$$

A Harder Example

Call the three integrals on the right above I_1, I_2, I_3 respectively.

- I_1 : $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|(+C_1)$.
- I_2 : $\displaystyle\int \frac{1}{(2x+1)^2}\,dx = -\frac{1}{2(2x+1)}(+C_2)$.
- I_3 : $\displaystyle\int \frac{1}{x-2}\,dx = \ln|x-2|(+C_3)$.

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

Learning outcomes for Section 1.4

At the end of this section you should

- Know the difference between a definite and indefinite integral and be able to explain it accurately and precisely.
- Be able to evaluate a range of definite and indefinite integrals using the following methods:
    - direct methods;
    - suitably chosen substitutions;
    - integration by parts;
    - partial fraction expansions.