Review of last week · example on "substitution" technique

Determine $\displaystyle\int_1^4 \frac{1}{x+\sqrt{x}}\,dx$

?? Try writing as a product?

$$\int_1^4 \frac{1}{\sqrt{x}\,(\sqrt{x}+1)}\,dx$$

Try. $u = \sqrt{x} + 1 = x^{1/2} + 1$

$du/dx = \frac{1}{2}x^{-1/2} = \frac{1}{2}\frac{1}{\sqrt{x}}$

$\Longrightarrow \quad 2\,du = \frac{1}{\sqrt{x}}\,dx$

Limits of integration: $x = 1 \rightarrow u = \sqrt{1} + 1 = 2$ $\quad x=4 \rightarrow u=3$

Our problem: $\displaystyle 2\int_{u=2}^{u=3} \frac{1}{u}\,du = 2\ln u \Big|_{u=2}^{u=3} = 2\ln 3 - 2\ln 2 = \boxed{2\ln\frac{3}{2}}$

In this section we discuss the technique of integration by parts, which is essentially a reversal of the product rule of differentiation.

Example 24

Find $\int \boxed{x \cos x}\, dx$.

Solution How could $x \cos x$ arise as a derivative?
Well, $\cos x$ is the derivative of $\sin x$. So, if you were differentiating $\boxed{x \sin x}$, you would get $x \cos x$ but according to the product rule you would also get another term, namely $\sin x$.

Conclusion: $\int x \cos x \, dx = x \sin x + \cos x + C.$

$$\frac{d}{dx}\left[x \sin x\right] \underset{\text{rule}}{\overset{\text{product}}{=}} \boxed{x \cos x} + \sin x$$

derivative of $-\cos x$

$$\frac{d}{dx}\left[x \sin x + \cos x\right] = \boxed{x \cos x} + \cancel{\sin x} - \cancel{\sin x}$$

What happened in this example was basically that the product rule was reversed. This process can be managed in general as follows. Recall from differential calculus that if u and v are expressions involving x, then

$$* \quad \boxed{(uv)' = u'v + uv'.} \quad *$$

Suppose we integrate both sides here with respect to x. We obtain

$$\int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx \Longrightarrow \boxed{uv = \int u'v\,dx + \int uv'\,dx.}$$

This can be rearranged to give the Integration by Parts Formula :

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx.}$$

Here is the first example again, handled according to this scheme.

Example 25

Use the integration by parts technique to determine $\int x \cos x \, dx$.

$\int uv'\,dx$

Solution: Write

$$\boxed{\begin{array}{l} u = x \\ u' = 1 \end{array}} \quad \begin{array}{l} v' = \cos x \\ v = \sin x \end{array}$$

alternative
$u = \cos x$ $\quad v' = x$
$u' = -\sin x$ $\quad v = \frac{x^2}{2}$

would end up with $\int \frac{x^2 \sin x}{2}\,dx$

Then

$$\begin{aligned} \int x \cos x \, dx &= \int uv' \, dx = \boxed{uv - \int u'v \, dx} \\ &= x \sin x - \int 1 \sin x \, dx \\ &= x \sin x + \cos x + C. \end{aligned}$$

uv

An antiderivative for $\ln x$

$(\ln x)' = \frac{1}{x}$

Example 26

Determine $\int 1 \ln x \, dx$.

$\int uv' \, dx = uv - \int u'v$

Solution: Let $\boxed{u = \ln x}$, $v' = 1$.
Then $u' = \frac{1}{x}$, $v = x$.

$$\begin{aligned}\int \underset{v'}{1} \underset{u}{\ln x} \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x \ln x - \int \frac{1}{x} x \, dx \\ &= x \ln x - x + C. \quad \leftarrow \text{check}\end{aligned}$$

$\int \ln x$

Note: This example shows that sometimes problems which are not obvious candidates for integration by parts can be attacked using this technique.

Sometimes two applications of the integration by parts formula are needed, as in the following example.

Example 27

Evaluate $\int x^2 e^x \, dx$.

Solution: Let $u = x^2$, $v' = e^x$. Then $u' = 2x$, $v = e^x$.

$$\begin{aligned} \boxed{\int x^2 e^x \, dx} &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x^2 e^x - \boxed{\int 2xe^x \, dx} \\ &= x^2 e^x - 2\boxed{\int xe^x \, dx}. \end{aligned}$$

Let $\boxed{I = \int xe^x \, dx.}$

Two rounds (continued)

Let $\mathcal{I} = \int xe^x \, dx$

To evaluate $\mathcal{I}$ apply the integration by parts formula a second time.

$$\begin{array}{ll} u = x & v' = e^x \\ u' = 1 & v = e^x. \end{array}$$

Then $I = \int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C$. Finally

$$\int \underbrace{x^2 e^x \, dx}_{= x^2e^x - 2\mathcal{I}} = x^2e^x - 2xe^x + 2e^x + C.$$

An Example of Another Type

The next example shows another mechanism by which a second application of the integration by parts formula can succeed where the first is not enough.

Example 28

Determine $\int e^x \cos x \, dx.$

Solution Let

$$\begin{array}{ll} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x. \end{array}$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

$\int e^x \cos x \, dx$ (continued)

For $\displaystyle\int e^x \sin x \, dx$: Let

$$\begin{aligned} u &= e^x & v' &= \sin x \\ u' &= e^x & v &= -\cos x. \end{aligned}$$

Then

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx,$$

and

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx \right) \\ \implies 2 \int e^x \cos x \, dx &= e^x \sin x + e^x \cos x + C \\ \implies \int e^x \cos x \, dx &= \frac{1}{2}(e^x \sin x + e^x \cos x) + C \end{aligned}$$

A Definite Integral

Example 29

Evaluate $\displaystyle\int_0^1 (x+3)e^{2x}\,dx$.

Solution: Write $u = x+3,\ v' = e^{2x};\qquad u' = 1,\ v = \frac{1}{2}e^{2x}$

$$
\begin{aligned}
\int_0^1 (x+3)e^{2x}\,dx &= \int uv'\,dx = (uv)|_0^1 - \int_0^1 u'v\,dx \\
&= \frac{x+3}{2}e^{2x}\Big|_0^1 - \frac{1}{2}\int_0^1 e^{2x}\,dx \\
&= \frac{x+3}{2}e^{2x}\Big|_0^1 - \frac{1}{2}\times\frac{1}{2}e^{2x}\Big|_0^1 \\
&= \frac{4}{2}e^2 - \frac{3}{2}e^0 - \frac{1}{4}e^2 + \frac{1}{4}e^0 = \frac{7}{4}e^2 - \frac{5}{4}.
\end{aligned}
$$