Powers of x

Example 17

Determine $\int x^n \, dx$

Important Note: We know that in order to calculate the derivative of an expression like x^n, we reduce the index by 1 to $n-1$, and we multiply by the constant n. So

$$\frac{d}{dx}x^n = nx^{n-1}$$

in general. To find an antiderivative of x^n we have to reverse this process. This means that the index increases by 1 to $n+1$ and we multiply by the constant $\frac{1}{n+1}$. So

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined).

The Integral of $\frac{1}{x}$

Suppose that $x > 0$ and $y = \ln x$. Recall this means (by definition) that $e^y = x$. Differentiating both sides of this equation (with respect to x) gives

$$e^y \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, and

$$\int \frac{1}{x}\,dx = \ln x + C, \text{ for } x > 0.$$

If $x < 0$, then

$$\int \frac{1}{x}\,dx = \ln |x| + C.$$

This latter formula applies for all $x \neq 0$.

A definite integral

Example 18

Determine $\int_0^{\pi} \sin x + \cos x \, dx$.

Solution: We need to write down *any* antiderivative of $\sin x + \cos x$ and evaluate it at the limits of integration :

$$\begin{aligned}\int_0^{\pi} \sin x + \cos x \, dx &= -\cos x + \sin x|_0^{\pi} \\ &= (-\cos \pi + \sin \pi) - (-\cos 0 + \sin 0) \\ &= -(-1) + 0 - (-1 + 0) = 2.\end{aligned}$$

Note: To determine $\cos \pi$, start at the point $(1, 0)$ and travel counter–clockwise along the the unit circle for a distance of π, arriving at the point $(-1, 0)$. The x-coordinate of the point you are at now is $\cos \pi$, and the y-coordinate is $\sin \pi$.

The Chain Rule of Differentation tells us that in order to differentiate the expression $\sin x^2$, we should regard this expression as sin("something") whose derivative (with respect to "something") is cos("something"), then multiply this expression by the derivative of the "something" with respect to x. Thus

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2.$$

Equivalently

$$\int 2x \cos x^2 \, dx = \sin x^2 + C.$$

In this section, through a series of examples, we consider how one might go about reversing the differentiation process to get from $2x \cos x^2$ back to $\sin x^2$.

How Substitution Works

Example 19

Determine $\int 2x\sqrt{x^2+1}\,dx$.

Solution Notice that the integrand involves both the expressions x^2+1 and $2x$. Note also that $2x$ is the derivative of x^2+1.

1. Introduce the notation u and set $u = x^2+1$.
2. Note $\frac{du}{dx} = 2x$; rewrite this as $du = 2x\,dx$.
3. Then

$$\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}(2x\,dx) = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

4. So

$$\int 2x\sqrt{x^2+1}\,dx = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$