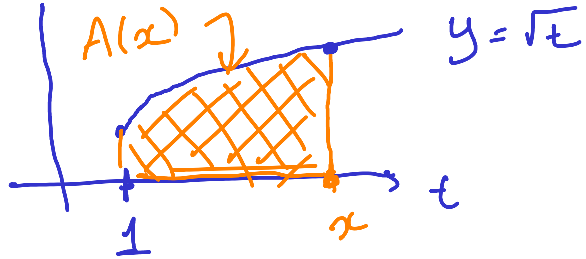
Lecture 23: Exam Advice

- Six questions, two hours (plus 30 minutes for upload via Blackboard assignment). Handwritten answers should be scanned or photographed, converted to pdf format (e.g. using Office Lens) and uploaded.
- Same arrangements for MA180, MA190, MA186.
- The three calculus questions will match the three chapters of the lecture notes and the six homework sheets.
- This year each of the calculus questions will have four parts all worth equal marks.
- Read the questions carefully. Make sure you answer exactly what is asked, as clearly as you can.
- Think about the learning outcomes when you are preparing.
- Think about communicating to the person who will read your script that you have achieved the learning outcomes.

(i) 

(a) Define a function A by

$$A(x) = \int_1^x \sqrt{t}\,dt$$

(ii) $A'(x) = \sqrt{x}$ for $x > 1$

(iii) $A'(4) = \sqrt{4} = 2$

for $x \geq 1$.

i Draw a diagram that indicates the meaning of $A(x)$.

ii. What does the Fundamental Theorem of Calculus say about the function A?

iii. What is $A'(4)$?

Evaluate the following integrals.

(b) $\displaystyle\int_1^2 xe^{3x}\,dx$ (c) $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$ (d) $\displaystyle\int \frac{x^2+3x+3}{x+1}\,dx$

(b) $\int_1^2 x e^{3x}\,dx$ Integration by parts $\int uv'\,dx = uv - \int u'v\,dx$

$u = x \qquad v' = e^{3x}$

$u' = 1 \qquad v = \frac{1}{3}e^{3x}$

$$\int_1^2 x e^{3x}\,dx = \frac{1}{3}xe^{3x}\Big|_1^2 - \frac{1}{3}\int_1^2 e^{3x}\,dx$$

$$= \frac{2}{3}e^6 - \frac{1}{3}e^3 - \frac{1}{9}e^{3x}\Big|_1^2$$

$$= \frac{2}{3}e^6 - \frac{1}{3}e^3 - \frac{1}{9}e^6 + \frac{1}{9}e^3$$

$$= \frac{5}{9}e^6 - \frac{2}{9}e^3$$

(c) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$ Substitute $u = \sqrt{x} = x^{1/2}$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad = \quad 2\,du = \frac{1}{\sqrt{x}}\,dx$$

Our integral becomes $2\int \cos u \, du$

$$2\int \cos u \, du = 2\sin u + C.$$
$$= 2\sin(\sqrt{x}) + C$$

(d) $$\int \frac{x^2+3x+3}{x+1}\,dx$$

$$\begin{array}{r|l}
 & x+2 \\
\hline
x+1 & x^2+3x+3 \\
 & x^2+x \\
\hline
 & 2x+3 \\
 & 2x+2 \\
\hline
 & 1
\end{array}
\implies \frac{x^2+3x+3}{x+1} = x+2+\frac{1}{x+1}$$

$$\int \left(x+2+\frac{1}{x+1}\right)dx = \frac{x^2}{2}+2x+\ln|x+1| + C$$

Sample Question 2

(a) State whether each of the following assertions is true or false. Give a short explanation (one or two lines) for your answer in each case.

- i. Every unbounded subset of $\mathbb{R}$ is infinite.
- ii Every infinite subset of $\mathbb{R}$ is unbounded.
- iii If a subset of $\mathbb{Q}$ is bounded above, then it has a maximum element.

(b) State what is means for an infinite set to be *countable*. Show that the set $\mathbb{Z}$ of integers is countable.

(c) Let $S = \left\{ \dfrac{2n+1}{n^2} : n \in \mathbb{Z},\ n \geq 1 \right\}$.

1. List four elements of S.
2. Show that S is bounded. Does S have a minimum element? Does S have a maximum element? Determine the infimum and supremum of S.

(d) Give an example of a bounded set of rational numbers whose infimum is irrational and whose supremum is rational.

Q2 (a)

(i) Every unbounded subset of $\mathbb{R}$ is infinite $\boxed{\text{TRUE}}$

If a subset of $\mathbb{R}$ is finite, it has a minimum and a maximum element, so it must be bounded.

(ii) Every infinite subset of $\mathbb{R}$ is unbounded $\boxed{\text{FALSE}}$

$(0,1)$ is an example of an infinite subset of $\mathbb{R}$ that is bounded.

(iii) If a subset of $\mathbb{Q}$ is bounded above, then it has a maximum element $\boxed{\text{FALSE}}$

$(0,1) \cap \mathbb{Q}$ is an example of a bounded subset of $\mathbb{Q}$ that has no maximum element.

(b) An infinite set S is <u>countable</u> if a bijective correspondence exists between S and the set $\mathbb{N}$ of natural numbers, or equivalently if the elements of S can be listed.

A bijective correspondence between $\mathbb{N}$ and $\mathbb{Z}$:

$\mathbb{N}$	1	2	3	4	5	6	7	8	9	10	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Z}$	0	1	-1	2	-2	3	-3	4	-4	5	...

(c) $$S = \left\{ \frac{2n+1}{n^2} : n \in \mathbb{Z},\ n \geq 1 \right\}$$

1. Four elements of S: $3, \ 5/4, \ 7/9, \ 9/16$

2. S is bounded below since all its elements are positive. Elements of S have the form
$$\frac{2n+1}{n^2} = \frac{2}{n} + \frac{1}{n^2}, \text{ for positive integers } n.$$

For $n \geq 1$, $\frac{2}{n}$ is bounded above by 2, and $1/n^2$ is bounded above by 1.

Thus 3 is an upper bound for S

3 is also the maximum element of S.

S does not have a minimum element. Its infimum is 0 (by choosing n great enough, we can arrange for $2/n + 1/n^2$ to be arbitrarily close to 0).

(d) $\{ x \in \mathbb{Q} : 2 \leq x^2 \leq 4 \text{ and } x > 0 \}$ is such a set.

Its infimum is $\sqrt{2}$ and its supremum is 2.

Sample Question 3

(a) Give an example of

i. a convergent sequence of real numbers; $1, 1, 1, 1, 1, \ldots$ converges to 1 or $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converges to 0

ii. a sequence of real numbers that is bounded and divergent; $0, 5, 0, 5, 0, 5, \ldots$

iii. a sequence of real numbers that is convergent and is not monotonic. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$

(b) A sequence (a_n) of real numbers is defined by

$$\boxed{a_0 = 4, \; \boxed{a_n = \frac{1}{2}(a_{n-1} - 2)} \text{ for } n \geq 1.}$$

i. Write down the first four terms of the sequence.

ii. Show that the sequence is bounded below by -2.

iii. Show that the sequence is monotonically decreasing.

iv. State why it can be deduced that the sequence is convergent, and determine its limit.

(i) $a_0 = 4$ $\quad a_1 = \frac{1}{2}(a_0 - 2) = \frac{1}{2}(4-2) = 1$

$a_2 = \frac{1}{2}(1-2) = -\frac{1}{2}$

$a_3 = \frac{1}{2}(-\frac{1}{2} - 2) = -\frac{5}{4}$

(c) Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(d) Find the first three terms in the Maclaurin series of $e^{\cos x}$.

(b) $a_0 = 4$ $\boxed{a_n = \frac{1}{2}(a_{n-1} - 2)}$ for $n \geq 1$

(ii) Suppose $a_{k-1} \geq -2$

Then $a_{k-1} - 2 \geq -4$

$$a_k = \frac{1}{2}(a_{k-1} - 2) \geq \frac{1}{2}(-4) = -2$$

$$\Rightarrow \quad a_k \geq -2$$

We know $a_0 \geq -2$. Then $a_1 \geq -2$, $a_2 \geq -2$, etc.
So all terms are ≥ -2.

(iii) Need $a_k \leq a_{k-1}$

$$a_{k-1} \geq -2 \quad \Rightarrow \quad -2 \leq a_{k-1}$$

$$a_{k-1} - 2 \leq a_{k-1} + a_{k-1} = 2a_{k-1}$$

$$a_k = \frac{1}{2}(a_{k-1} - 2) \leq \frac{1}{2}(2a_{k-1}) = a_{k-1}$$

(iv) Our sequence is bounded below and decreasing $\Rightarrow$ convergent by the monotone convergent theorem.

The limit L must satisfy

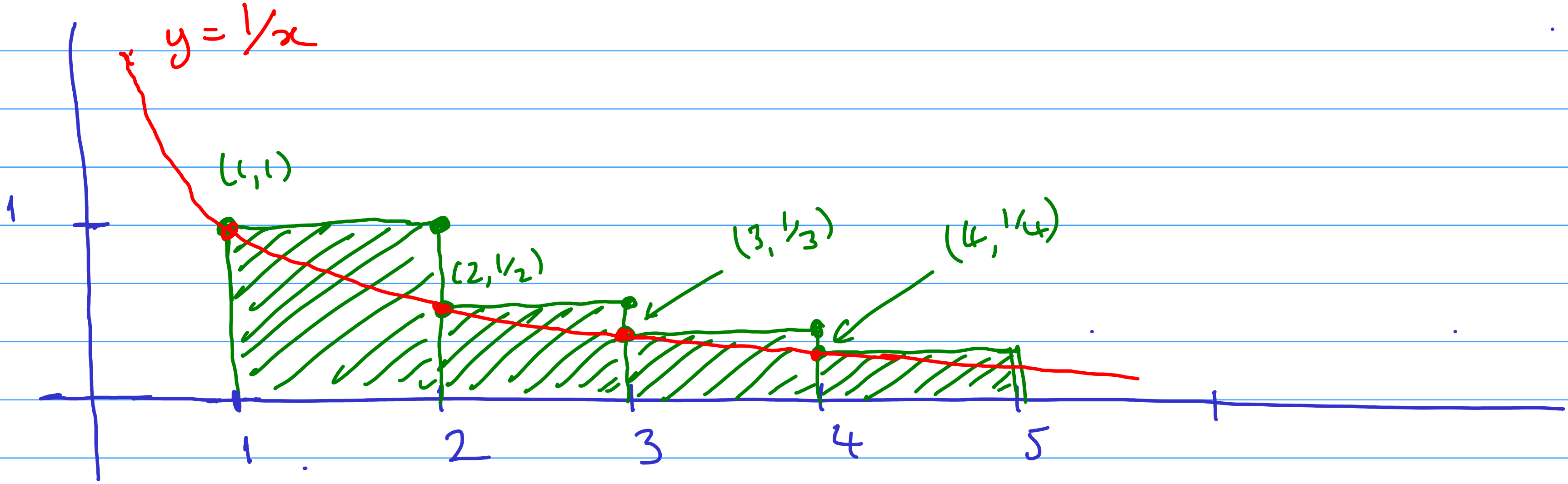
$$\left[a_n = \frac{1}{2}(a_{n-1} - 2) \right.$$

$$L = \frac{1}{2}(L-2)$$

$$2L = L - 2$$

$$\boxed{L = -2}$$

(c) We represent y_n (for $n \geq 1$) as the area of the rectangle of height y_n standing on the interval $(n, n+1)$ of the x-axis. The sum of the harmonic series is the total area of these rectangles, shaded in green in this picture



The area shaded green exceeds the area enclosed under the graph $y = 1/x$, to the right of $x = 1$, which is represented by the improper integral: $\int_1^\infty \frac{1}{x}\,dx$. This improper integral is divergent, the corresponding area is infinite, and hence the total area of the green rectangles is also infinite, and the harmonic series diverges.

(d) Write $f(x) = e^{\cos x}$

$$f(0) = e^{\cos(0)} = e^{1} = \boxed{e}$$

$$f'(x) = -\sin x \, e^{\cos x} \Rightarrow f'(0) = -\sin(0)\, e^{\cos(0)} = \boxed{0}$$

$$f''(x) = \sin^2 x \, e^{\cos x} - \cos x \, e^{\cos x} \Rightarrow f''(0) = 0 - 1e^{1} = -e$$

$$f''(0)/2! = \boxed{-e/2}$$

First 3 terms of Maclaurin series: e, $0x$, $-e/2\, x^2$

(The Maclaurin series begins $e + 0x - e/2\, x^2$)