Section 3.3: Introduction to Infinite Series

Definition 69

A series or infinite series is the sum of all the terms in a sequence.

Example 70 (Examples of infinite series)

1. $\boxed{\sum_{n=1}^{\infty}} n = 1 + 2 + 3 + \dots$
2. *A geometric series*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

Every term in this series is obtained from the previous one by multiplying by the common ratio $\frac{1}{2}$. This is what geometric means.

Examples of Series (continued)

Example 71

3. *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

4. *An alternating series*

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

Notes

1. For now these infinite sums are just formal expressions or arrangements of symbols. Whether it is meaningful to think of them as numbers or not is something that can be investigated. $\sum_{n=1}^{\infty} n = 1+2+3+\ldots$
2. A series is not the same thing as a sequence - don't confuse these terms! A sequence is a list of numbers. A series is an infinite sum.
3. The "sigma" notation for sums: sigma (lower case σ, upper case Σ) is a letter from the Greek alphabet, the upper case $\sum$ is used to denote sums. The notation $\sum_{n=i}^{j} a_n$ means: i and j are integers and $i \leq j$. For each n from i to j the number a_n is defined; the expression above means the sum of the numbers a_n where n runs through all the values from i to j, i.e.

$$\sum_{n=i}^{j} a_n = a_i + a_{i+1} + a_{i+2} + \cdots + a_{j-1} + a_j.$$

$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots$

For infinite sums we can have $-\infty$ and/or ∞ (instead of fixed integers i and j) as subscripts and superscripts for the summation.

In the examples above we can start from the beginning, adding terms at the start of the series. Adding term by term we get the following lists.

1. $$\boxed{\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots}$$

   $1,\ 1+2,\ 1+2+3,\ 1+2+3+4,\ 1+2+3+4+5, \dots$ $1, 3, 6, 10, 15, \dots$

   Since the terms being added on at each stage are getting bigger, the numbers in the list above will keep growing (faster and faster as n increases) - we can't associate a numerical value with this infinite sum.

2. A geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

Sequence converging to 2.

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3} \dots \qquad 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32} \dots$$

In this example the terms that are being added on at each step ($\frac{1}{2^n}$) are getting smaller and smaller as n increases, and the numbers in the list appear to be converging to 2.

3. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \dots \qquad 1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$$

It is harder to see what is going on here.

Notes

4. An alternating series

$$\sum_{n=0}^{\infty}(-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

$1,\ 1-1,\ 1-1+1,\ 1-1+1-1,\ 1-1+1-1+1\dots \quad 1, 0, 1, 0, 1, \dots$

The terms being "added on" at each step are alternating between 1 and -1, and as we proceed with the summation the "running total" alternates between 0 and 1. There is no numerical value that we can associate with the infinite sum $\sum_{n=0}^{\infty}(-1)^n$.

Note: The series in 2. above converges to 2, the series in 1. and 4. are both divergent and it is not obvious yet but the series in 3. is divergent as well. Our next task is to give precise meanings to these terms for series. In order to do this we need some terminology. We know what it means for a sequence to converge, but we don't yet have a definition of convergence for series.

Definition 72

For a series $\sum_{n=1}^{\infty} a_n$, and for $k \geq 1$, let

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$$\boxed{s_k} = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

The s_k are numbers

Thus $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$ etc.
Then s_k is called the *kth partial sum* of the series, and the sequence $\{s_k\}_{k=1}^{\infty}$ is called the *sequence of partial sums* of the series.
If the sequence of partial sums converges to a limit s, the series is said to *converge* and s is called its sum. In this situation we can write $\sum_{n=1}^{\infty} a_n = s$. If the sequence of partial sums *diverges*, the series is said to *diverge*.

Recall Example 2 above:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

In this example, for $k \geq 0$,

$$s_k = \sum_{n=0}^{k} \frac{1}{2^n} = 1 + \cancel{\frac{1}{2}} + \cancel{\frac{1}{4}} + \dots \cancel{\frac{1}{2^k}}$$

$$\boxed{\frac{1}{2} s_k} = \sum_{n=1}^{k} \frac{1}{2^{n+1}} = \cancel{\frac{1}{2}} + \cancel{\frac{1}{4}} + \dots \cancel{\frac{1}{2^k}} + \boxed{\frac{1}{2^{k+1}}}$$

Then

$$s_k - \frac{1}{2} s_k = \frac{1}{2} s_k = 1 - \frac{1}{2^{k+1}} \implies s_k = 2 - \frac{1}{2^k}.$$

So the sequence of partial sums has kth term $\boxed{2 - \frac{1}{2^k}.}$ This sequence converges to 2 so the series converges to 2.

General geometric series

Consider the sequence of partial sums for the geometric series

$$\boxed{\sum_{n=0}^{\infty} ar^n} = a + ar + ar^2 + \dots$$

Common ratio r

Initial term a

(This is a geometric series with initial term a and common ratio r.) The kth partial sum s_k is given by

$$\begin{array}{lllllllllll} s_k & = & \sum_{n=0}^{k} ar^n & = & a + & \cancel{ar} + & \dots & + & & \cancel{ar^k} & \\ rs_k & = & \sum_{n=0}^{k} ar^{n+1} & = & & \cancel{ar} + & \cancel{ar^2} + & \dots & + & \cancel{ar^k} & + \; ar^{k+1} \end{array}$$

Then $\boxed{(1-r)s_k = a - ar^{k+1}} \Longrightarrow \boxed{s_k = \boxed{\frac{a(1-r^{k+1})}{1-r}}}$. If $|r| < 1$, then $r^{k+1} \to 0$ as $k \to \infty$, and the sequence of partial sums (hence the series) converges to $\frac{a}{1-r}$. If $|r| \geq 1$ the series is divergent.

The harmonic series is divergent

Theorem 73

The harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

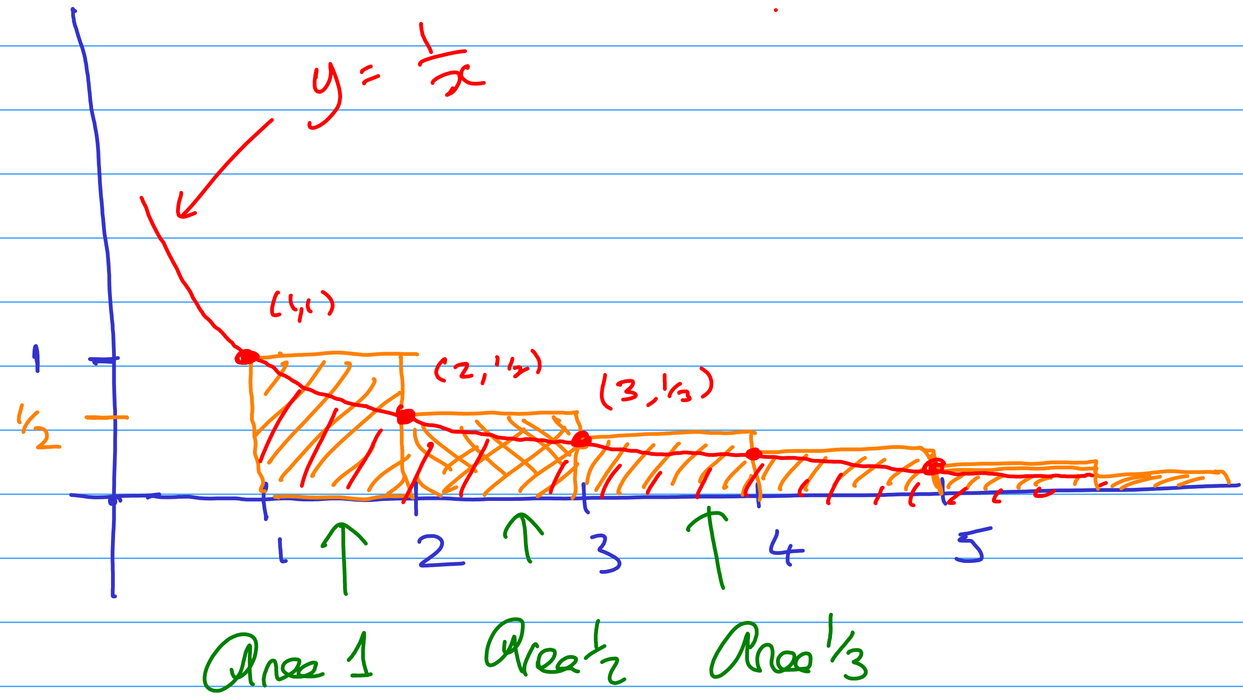
Proof: Think of $\frac{1}{n}$ as the area of a rectangle of height $\frac{1}{n}$ and width 1, sitting on the interval $[n, n+1]$ on the x-axis. So the $\frac{1}{1}$ corresponds to a square of area 1 sitting on the interval $[1, 2]$, the term $\frac{1}{2}$ corresponds to a rectangle of area $\frac{1}{2}$ sitting on the interval $[2, 3]$ and so on.
The total area accounted for by these triangles is the sum of the harmonic series, and this exceeds the area accounted for by the improper integral

$$\int_1^{\infty} \frac{1}{x}\, dx.$$

From Section 1.5 we know that this area is infinite.

Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Draw a rectangle of height $\frac{1}{n}$ with base $[n, n+1]$, for $n \geq 1$



We know that the area enclosed by $y = 1/x$ and the X-axis, from $x=1$ onwards, is infinite

Section 1.5 $\int_1^{\infty} \frac{1}{x}\, dx$ is divergent

A necessary condition for convergence

Note: A necessary condition for the series

$$\sum_{n=1}^{\infty} a_n$$

to converge is that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0; i.e. that $a_n \to 0$ as $n \to \infty$. If this does *not* happen, then the sequence of partial sums has no possibility of converging.

The example of the harmonic series shows that the condition $a_n \to 0$ as $n \to \infty$ is not sufficient to guarantee that the series $\sum_{n=1}^{\infty}$ will converge.

After studying this section you should be able to

- explain what an infinite series is and what it means for an infinite series to converge;
- Give examples of convergent and divergent series;
- show that the harmonic series is divergent;
- Use the "sigma" notation for sums.