Bounded Sequences

As for subsets of $\mathbb{R}$, there is a concept of boundedness for sequences. Basically a sequence is bounded (or bounded above or bounded below) if the set of its terms, considered as a subset of $\mathbb{R}$, is bounded (or bounded above or bounded below). More precisely :

Definition 62

The sequence (a_n) is bounded above if there exists a real number M for which $a_n \leq M$ for all $n \in \mathbb{N}$.
The sequence (a_n) is bounded below if there exists a real number m for which $m \leq a_n$ for all $n \in \mathbb{N}$.
The sequence (a_n) is bounded if it is bounded both above and below.

Example 63

The sequence (n) is bounded below (for example by 0) but not above.
The sequence $(\sin n)$ is bounded below (for example by -1) and above (for example by 1).

Convergent $\implies$ Bounded

Theorem 64

If a sequence is convergent it must be bounded.

Proof Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence with limit L.

Then (by definition of convergence) there exists a natural number N such that every term of the sequence after a_N is between $L-1$ and $L+1$.

The set consisting of the first N terms of the sequence is a finite set : it has a maximum element M_1 and a minimum element m_1.

Let $M = \max\{M_1, L+1\}$ and let $m = \min\{m_1, L-1\}$. Then (a_n) is bounded above by M and bounded below by m.

So our sequence is bounded.

Increasing and decreasing sequences

Definition 65

A sequence (a_n) is called increasing if $a_n \leq a_{n+1}$ for all $n \geq 1$.
A sequence (a_n) is called strictly increasing if $a_n < a_{n+1}$ for all $n \geq 1$.
A sequence (a_n) is called decreasing if $a_n \geq a_{n+1}$ for all $n \geq 1$.
A sequence (a_n) is called strictly decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.

Definition 66

A sequence is called monotonic if it is either increasing or decreasing.
Similar terms: monotonic increasing, monotonic decreasing, monotonically increasing/decreasing.

Note: These definitions are not *entirely* standard. Some authors use the term *increasing* for what we have called *strictly increasing* and/or use the term *nondecreasing* for what we have called *increasing*.

Examples

1. An increasing sequence is bounded below but need not be bounded above. For example
$$(n)_{n=1}^{\infty} : 1,\ 2,\ 3,\ \dots$$
2. A bounded sequence need not be monotonic. For example
$$((-1)^n) : -1,\ 1,\ -1,\ 1,\ -1,\ \dots$$
3. A convergent sequence need not be monotonic. For example
$$\left(\frac{(-1)^{n+1}}{n}\right)_{n=1}^{\infty} : 1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \dots$$
4. A monotonic sequence need not be convergent, as Example 1 above shows.

However, if a sequence is bounded and monotonic, it is convergent. This is the Monotone Convergence Theorem.

The Monotone Convergence Theorem

Theorem 67

If a sequence $(a_n)_{n=1}^{\infty}$ is montonic and bounded, then it is convergent.

Proof: Suppose that (a_n) is increasing and bounded.

Then the set $\{a_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$ and by the Axiom of Completeness it has a least upper bound (or supremum) L.

We will show that the sequence (a_n) converges to L.

Choose a (very small) $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound for $\{a_n : n \in \mathbb{N}\}$, becasue L is the least upper bound.

This means there is some $N \in \mathbb{N}$ for which $L - \varepsilon < a_N$. Since L is an upper bound for $\{a_n : n \in \mathbb{N}\}$, this means

$$L - \varepsilon < a_N \leq L$$

Proof of the Monotone Convergence Theorem (continued)

$$L - \varepsilon < a_N \leq L$$

Since the sequence (a_n) is increasing and its terms are bounded above by L, every term after a_N is between a_N and L, and therefore between $L - \varepsilon$ and L. These terms are all within ε of L

Using the fact that our sequence is increasing and bounded, we have

- Identified L as the least upper bound for the set of terms in our sequence
- Showed that no matter how small an ε we take, there is a point in our sequence beyond which all terms are within ε of L.

This is exactly what it means for the sequence to converge to L.

An Example (from 2015 Summer Exam)

Example 68

A sequence (a_n) *of real numbers is defined by*

$$a_0 = 4, \; a_n = \frac{1}{2}(a_{n-1} - 2) \text{ for } n \geq 1.$$

1. *Write down the first four terms of the sequence.*
2. *Show that the sequence is bounded below.*
3. *Show that the sequence is montonically decreasing.*
4. *State why it can be deduced that the sequence is convergent, and determine its limit.*

Note: This is an example of a sequence that is defined recursively. This means that the first term is given and subsequent terms are defined (one by one) in terms of previous ones. We are not given a general formula for the nth term although one may exist.

Learning Outcomes for Section 3.2

After studying this section you should be able to

- Explain what a sequence is;
- State what it means for a sequence to be
  - convergent or divergent;
  - bounded or unbounded (above or below);
  - monotonic, increasing or decreasing.
- Give and/or identify examples of sequences with or without various properties (or combinations of properties) from the above list;
- State, prove and apply the Monotone Convergence Theorem;
- Analyze examples similar to Example 83.

Definition 69

A series or infinite series is the sum of all the terms in a sequence.

Example 70 (Examples of infinite series)

1. $\displaystyle\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots$
2. *A geometric series*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots$$

Every term in this series is obtained from the previous one by multiplying by the common ratio $\frac{1}{2}$. This is what geometric means.

Examples of Series (continued)

Example 71

3. *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

4. *An alternating series*

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

Notes

1. For now these infinite sums are just formal expressions or arrangements of symbols. Whether it is meaningful to think of them as numbers or not is something that can be investigated.
2. A series is not the same thing as a sequence - don't confuse these terms! A sequence is a list of numbers. A series is an infinite sum.
3. The "sigma" notation for sums: sigma (lower case σ, upper case Σ) is a letter from the Greek alphabet, the upper case $\sum$ is used to denote sums. The notation $\sum_{n=i}^{j} a_n$ means:
   i and j are integers and $i \leq j$. For each n from i to j the number a_n is defined; the expression above means the sum of the numbers a_n where n runs through all the values from i to j, i.e.

$$\sum_{n=i}^{j} a_n = a_i + a_{i+1} + a_{i+2} + \cdots + a_{j-1} + a_j.$$

   For infinite sums we can have $-\infty$ and/or ∞ (instead of fixed integers i and j) as subscripts and superscripts for the summation.

Sequences of partial sums

In the examples above we can start from the beginning, adding terms at the start of the series. Adding term by term we get the following lists.

1. $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$

   $1, 1+2, 1+2+3, 1+2+3+4, 1+2+3+4+5, \dots$ $1, 3, 6, 10, 15, \dots$

   Since the terms being added on at each stage are getting bigger, the numbers in the list above will keep growing (faster and faster as n increases) - we can't associate a numerical value with this infinite sum.

Examples (continued)

2. A geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3} \dots \quad 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32} \dots$$

In this example the terms that are being added on at each step ($\frac{1}{2^n}$) are getting smaller and smaller as n increases, and the numbers in the list appear to be converging to 2.

3. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \dots \quad 1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$$

It is harder to see what is going on here.

Notes

4. An alternating series

$$\sum_{n=0}^{\infty}(-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

$1,\ 1-1,\ 1-1+1,\ 1-1+1-1,\ 1-1+1-1+1\dots \quad 1, 0, 1, 0, 1, \dots$

The terms being "added on" at each step are alternating between 1 and -1, and as we proceed with the summation the "running total" alternates between 0 and 1. There is no numerical value that we can associate with the infinite sum $\sum_{n=0}^{\infty}(-1)^n$.

Note: The series in 2. above converges to 2, the series in 1. and 4. are both divergent and it is not obvious yet but the series in 3. is divergent as well. Our next task is to give precise meanings to these terms for series. In order to do this we need some terminology. We know what it means for a sequence to converge, but we don't yet have a definition of convergence for series.