Section 1.2: The Definite Integral

Goal of this section:

In the last section we concluded that we would like a theory for discussing (and hopefully calculating) areas enclosed between the graphs of known functions and the x-axis, within specified intervals.

Such a theory does exist and it forms a large part of what is called integral calculus.

In order to develop and use this theory we need a technical language and notation for talking about areas under curves.
The goal of this section is to understand this notation and be able to use it.

An Example

Example 7

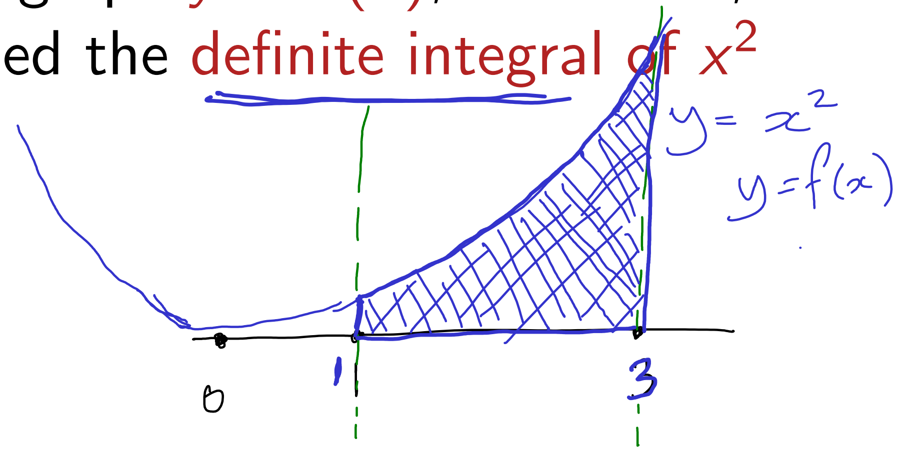
Suppose that f is the function defined by $f(x) = x^2$.

Note that $f(x)$ is positive when $1 \leq x \leq 3$.

This means that in the region between the vertical lines $x = 1$ and $x = 3$, the graph $y = f(x)$ lies completely above the x-axis.

The area of the region enclosed between the graph $y = f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 3$ is called the definite integral of x^2 from $x = 1$ to $x = 3$, and denoted by

$$\boxed{\int_1^3 x^2\, dx.}$$



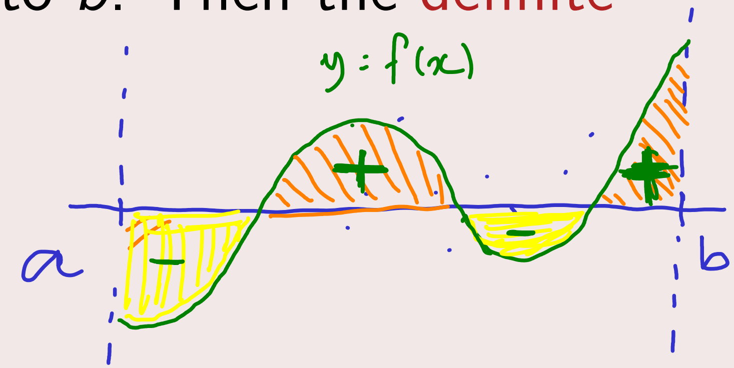
Note that in general for a function f and numbers a and b, the definite integral $\boxed{\int_a^b f(x)\, dx}$ is a number.

Definition of the definite integral

Definition 8

Let a and b be fixed real numbers, with $a < b$. Let f be a function for which it makes sense to talk about the area enclosed between the graph of f and the x-axis, over the interval from a to b. Then the definite integral from a to b of f, denoted

$$\int_a^b f(x)\,dx$$

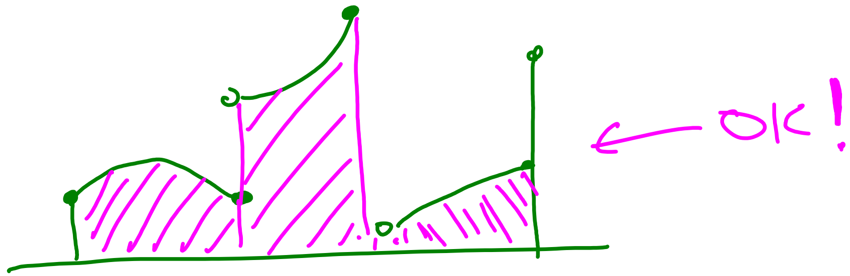


is defined to be the number obtained by subtracting the area enclosed below the x-axis by the graph $y = f(x)$ and the vertical lines $x = a$ and $x = b$ from the area enclosed above the x-axis by the graph $y = f(x)$ and the vertical lines $x = a$ and $x = b$.

So the definite integral is essentially the area enclosed between the graph and the x-axis in the relevant interval, except that area below the x-axis is considered to be negative.

Notes

1. In our definition, what is meant by the phrase "for which it makes sense to talk about the area enclosed between the graph of f and the x-axis" is (more or less) that the graph $y = f(x)$ is not just a scattering of points, but consists of a curve or perhaps more than one curve. There is a formal theory about "integrable functions" that makes this notion precise.



2. Note on Notation

The notation surrounding definite integrals is a bit unusual. This note explains the various components involved in the expression

$$\boxed{\int_a^b \underline{f(x)}\, \underline{dx}.}$$

- "$\int$" is the integral sign.
- The "dx" indicates that f is a function of the variable x, and that we are talking about area between the graph of $f(x)$ against x and the x-axis.
- The "$f(x)$" in $\int_a^b f(x)\,dx$ is called the integrand. It is the function whose graph is the upper (or lower) boundary of the region whose area is being described.

Notes (continued)

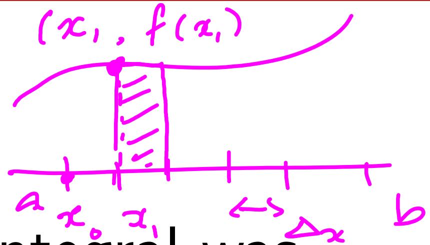
$$\int_a^b f(x)\,dx$$

- The numbers a and b are respectively called the lower and upper (or left and right) limits of integration. They determine the left and right boundaries of the region whose area is being described.
  In the expression $\int_a^b f(x)\,dx$, the limits of integration a and b are taken to be values of the variable x - this is included in what is to be interpreted from "dx".
  If there is any danger of ambiguity about this, you can write

$$\int_{x=a}^{x=b} f(x)\,dx \quad \text{instead of} \quad \int_a^b f(x)\,dx.$$

  Please do not confuse this use of the word "limit" with its other uses in calculus.

Some Historical Remarks



The notation that is currently in use for the definite integral was introduced by Gottfried Leibniz around 1675. The rationale for it is as follows :

$$\sum_i f(x_i)\,\Delta x$$

- Areas were estimated as we did in Section 3.1. The interval from a to b would be divided into narrow subintervals, each of width $\boxed{\Delta x}$.
- The name x_i would be given to the left endpoint of the ith subinterval, and the height of the graph above the point x_i would be $f(x_i)$.
- So the area under the graph on this ith subinterval would be approximated by that of a rectangle of width Δx and height $f(x_i)$.

$$\int_a^b f(x)\,dx$$

Some Historical Remarks (continued)

- The total area would be approximated by the sum of the areas of all of these narrow rectangles, which was written as

$$\sum f(x_i)\Delta x.$$

  The accuracy of this estimate improves as the width of the subintervals gets smaller and the number of them gets larger; the true area is the limit of this process as $\Delta x \to 0$.
- The notation "dx" was introduced as an expression to replace Δx in this limit, and the integral sign $\int$ is a "limit version" of the summation sign $\sum$. The integral symbol itself is based on the "long s" character which was in use in English typography until about 1800.

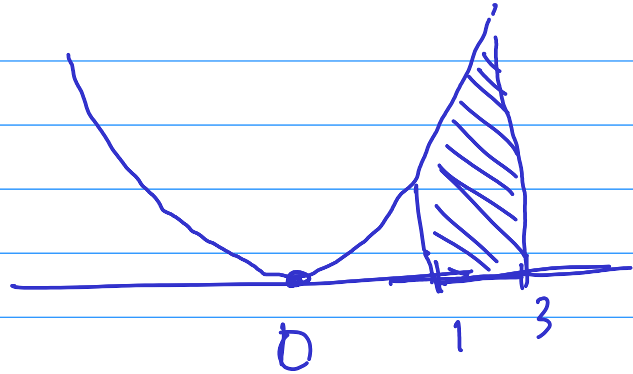
For more information on the history of calculus and of mathematics generally, see `http://www-history.mcs.st-and.ac.uk/index.html`.

Learning Outcome for Section 1.2

Just to be able to use the notation for definite integrals correctly.
This notation is admittedly a bit obscure but with careful attention you can certainly get it right!

Back to first example

$$\int_1^3 x^2 \, dx$$



To calculate this

Find something whose derivative is x^2.

$$\frac{x^3}{3}$$

$$\left[\frac{d}{dx}\left(\frac{x^3}{3}\right) = \frac{3x^2}{3} = x^2\right]$$

Subtract the value at 1 from the value at 3

$$\frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3} \longleftarrow \int_1^3 x^2 \, dx = 26/3$$