

1.4.3 Partial Fraction Expansions - Integrating Rational Functions

We know how to integrate polynomial functions; for example

$$\int 2x^2 + 3x - 4 \, dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x + C.$$

We also know that

$$\int \frac{1}{x} \, dx = \ln|x| + C.$$

and that

$$\int \frac{1}{x^n} \, dx = -\frac{1}{n-1} \frac{1}{x^{n-1}} + C,$$

for $n > 1$.

This section is about integrating *rational functions*; i.e. quotients in which the numerator and denominator are both polynomials.

REMARK: If we were presented with the task of adding the expressions $\frac{2}{x+3}$ and $\frac{1}{x+4}$, we would take $(x+3)(x+4)$ as a *common denominator* and write

$$\frac{2}{x+3} + \frac{1}{x+4} = \frac{2(x+4)}{(x+3)(x+4)} + \frac{1(x+3)}{(x+3)(x+4)} = \frac{2(x+4) + 1(x+3)}{(x+3)(x+4)} = \frac{3x+11}{(x+3)(x+4)}.$$

Question: Suppose we were presented with the expression $\frac{3x+11}{(x+3)(x+4)}$ and asked to rewrite it in the form $\frac{A}{x+3} + \frac{B}{x+4}$, for *numbers* A and B. How would we do it?

Another Question Why would we want to do such a thing?

Answer to the second question: Maybe if we want to integrate the expression: we know how to integrate things like $\frac{1}{x+3}$, but not things like $\frac{3x+11}{(x+3)(x+4)}$.

Answer to the first question: Write

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}.$$

Then

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A(x+4)}{(x+3)(x+4)} + \frac{B(x+3)}{(x+3)(x+4)} = \frac{(A+B)x + 4A + 3B}{(x+3)(x+4)}.$$

This means $3x+11 = (A+B)x + 4A + 3B$ for all x , which means

$$A + B = 3, \text{ and } 4A + 3B = 11.$$

Thus $-4A - 4B = -12$, $-B = -1$, $B = 1$ and $A = 2$. So

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

Alternative Method: We want

$$3x + 11 = A(x+4) + B(x+3),$$

for *all* real numbers x . If this statement is true for all x , then in particular it is true when $x = -4$. Setting $x = -4$ gives

$$-12 + 11 = A(0) + B(-1) \implies B = 1.$$

Setting $x = -3$ gives

$$-9 + 11 = A(1) + B(0) \implies A = 2.$$

Thus

$$\frac{3x + 11}{(x + 3)(x + 4)} = \frac{2}{x + 3} + \frac{1}{x + 4}.$$

Expansions of rational functions of this sort are called *partial fraction expansions*.

Example 1.4.25. Determine $\int \frac{3x + 11}{(x + 3)(x + 4)} dx$.

SOLUTION : Write

$$\int \frac{3x + 11}{(x + 3)(x + 4)} dx = \int \frac{2}{x + 3} dx + \int \frac{1}{x + 4} dx$$

Then

$$\int \frac{3x + 11}{(x + 3)(x + 4)} dx = 2 \ln|x + 3| + \ln|x + 4| + C = \ln(x + 3)^2 + \ln|x + 4| + C.$$

Example 1.4.26. Determine $\int \frac{1}{x^2 + 5x + 6} dx$.

SOLUTION: Write $\frac{1}{x^2 + 5x + 6} = \frac{1}{(x + 2)(x + 3)}$ in the form

$$\frac{A}{x + 2} + \frac{B}{x + 3},$$

for constants A and B . This means

$$\frac{1}{(x + 2)(x + 3)} = \frac{A(x + 3) + B(x + 2)}{(x + 2)(x + 3)},$$

i.e. $1 = A(x + 3) + B(x + 2)$ for all x .

Thus

$$0x + 1 = (A + B)x + (3A + 2B),$$

which means $A + B = 0$ and $3A + 2B = 1$. This pair of equations has the unique solution $A = 1$, $B = -1$. Thus

$$\begin{aligned} \frac{1}{(x + 2)(x + 3)} &= \frac{1}{x + 2} - \frac{1}{x + 3} \\ \implies \int \frac{1}{(x + 2)(x + 3)} &= \int \frac{1}{x + 2} - \frac{1}{x + 3} dx \\ &= \ln|x + 3| - \ln|x + 2| + C. \end{aligned}$$

NOTE: Any expression of the form $\frac{f(x)}{g(x)}$ where

1. $f(x)$ and $g(x)$ are polynomials and $g(x)$ has higher degree than $f(x)$, and
2. $g(x)$ can be factorized as the product of *distinct* linear factors

$$g(x) = (x - a_1)(x - a_2) \dots (x - a_k)$$

has a *partial fraction expansion* of the form

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_k}{x - a_k},$$

where A_1, A_2, \dots, A_k are numbers.

Example 1.4.27. Determine $\int \frac{x^3 + 3x + 2}{x + 1} dx$.

In this example the degree of the numerator exceeds the degree of the denominator, so first apply long division to find the quotient and remainder upon dividing $x^3 + 3x + 2$ by $x + 1$.

We find that the quotient is $x^2 - x + 4$ and the remainder is -2 . Hence

$$\frac{x^3 + 3x + 2}{x + 1} = x^2 - x + 4 + \frac{-2}{x + 1}.$$

Thus

$$\int \frac{x^3 + 3x + 2}{x + 1} dx = \int x^2 - x + 4 dx - 2 \int \frac{1}{x + 1} dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x - 2 \ln|x + 1| + C.$$

NOTE: In the above example we had $\frac{f(x)}{g(x)}$ with $f(x)$ of greater degree than $g(x)$. In such cases we can always write

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)},$$

where the polynomials $q(x)$ and $r(x)$ are the quotient and remainder respectively on dividing $f(x)$ by $g(x)$, and the degree of $r(x)$ is less than that of $g(x)$.

Example 1.4.28. Determine $\int \frac{x + 1}{(2x + 1)^2(x - 2)} dx$.

In this case the denominator has a repeated linear factor $2x + 1$. It is necessary to include both $\frac{A}{2x + 1}$ and $\frac{B}{(2x + 1)^2}$ in the partial fraction expansion. We have

$$\frac{x + 1}{(2x + 1)^2(x - 2)} = \frac{A}{2x + 1} + \frac{B}{(2x + 1)^2} + \frac{C}{x - 2}.$$

Then

$$\frac{x + 1}{(2x + 1)^2(x - 2)} = \frac{A(2x + 1)(x - 2) + B(x - 2) + C(2x + 1)^2}{(2x + 1)^2(x - 2)}.$$

This means that the polynomials $x + 1$ and $A(2x + 1)(x - 2) + B(x - 2) + C(2x + 1)^2$ are equal, and therefore have the same value when x is replaced by any real number.

$$x = 2: \quad 3 = C(5)^2 \quad C = \frac{3}{25}$$

$$x = -\frac{1}{2}: \quad \frac{1}{2} = B\left(-\frac{5}{2}\right) \quad B = -\frac{1}{5}$$

$$x = 0: \quad 1 = A(1)(-2) + B(-2) + C(1)^2 \quad A = -\frac{6}{25}$$

Thus

$$\frac{x + 1}{(2x + 1)^2(x - 2)} = \frac{-6/25}{2x + 1} + \frac{-1/5}{(2x + 1)^2} + \frac{3/25}{x - 2}$$

and

$$\int \frac{x + 1}{(2x + 1)^2(x - 2)} dx = -\frac{6}{25} \int \frac{1}{2x + 1} dx - \frac{1}{5} \int \frac{1}{(2x + 1)^2} dx + \frac{3}{25} \int \frac{1}{x - 2} dx.$$

Call the three integrals on the right above I_1, I_2, I_3 respectively.

- I_1 : Let $u = 2x + 1$, $du = 2dx$, $dx = \frac{1}{2}du$.

$$\int \frac{1}{2x+1} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| (+C_1) = \frac{1}{2} \ln |2x + 1| (+C_1).$$

- I_2 : Let $u = 2x + 1$, $du = 2dx$, $dx = \frac{1}{2}du$.

$$\int \frac{1}{(2x+1)^2} dx = \frac{1}{2} \int u^{-2} du = -\frac{1}{2}u^{-1} (+C_2) = -\frac{1}{2(2x+1)} (+C_2).$$

- I_3 : $\int \frac{1}{x-2} dx = \ln |x - 2| (+C_3).$

Thus

$$\int \frac{x + 1}{(2x + 1)^2(x - 2)} dx = -\frac{3}{25} \ln |2x + 1| + \frac{1}{10(2x + 1)} + \frac{3}{25} \ln |x - 2| + C.$$

LEARNING OUTCOMES FOR SECTION 1.4

At the end of this section you should

- Know the difference between a definite and indefinite integral and be able to explain it accurately and precisely.
- Be able to evaluate a range of definite and indefinite integrals using the following methods:
 - direct methods;
 - suitably chosen substitutions;
 - integration by parts;
 - partial fraction expansions.