A Sequence

Anja, Bertha, Collette, Deirdre and Eva are considering the sequence $(a_n)_{n\geq 1}$ defined by

$$a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ \frac{1}{2n} & \text{if } n \text{ is even.} \end{cases}$$

They are trying to determine whether this sequence is convergent or not.

$a_n = \frac{1}{n}$ for n odd, $\frac{1}{2n}$ for n even.

Anja writes:
The terms of the sequence go like this:

$$1, \frac{1}{4}, \frac{1}{3}, \frac{1}{8}, \frac{1}{5}, \frac{1}{10}, \dots$$

The sequence is bounded and decreasing so it is convergent by the Monotone Convergence Theorem.

$a_n = \frac{1}{n}$ for n odd, $\frac{1}{2n}$ for n even.

Bertha writes:
The nth term of the sequence is either $\frac{1}{n}$ or $\frac{1}{2n}$ depending on whether n is odd or even. Both of these are approaching zero as n goes to infinity, so the sequence is convergent and its limit is zero.

$a_n = \frac{1}{n}$ for n odd, $\frac{1}{2n}$ for n even.

Collette writes:

The terms of the sequence go like this:

$$1, \frac{1}{4}, \frac{1}{3}, \frac{1}{8}, \frac{1}{5}, \frac{1}{10}, \dots$$

This sequence is not monotonic - it alternately increases and decreases, so although it is bounded we cannot apply the Monotone Convergence Theorem. We have no way to decide if the sequence is convergent or not.

$a_n = \frac{1}{n}$ for n odd, $\frac{1}{2n}$ for n even.

Deirdre writes:

Let ε be a very small positive number. No matter how small ε is, we can choose N big enough that both $\frac{1}{n}$ and $\frac{1}{2n}$ are less than ε whenever $n > N$. This proves that the sequence converges to zero.

$a_n = \frac{1}{n}$ for n odd, $\frac{1}{2n}$ for n even.

Eva writes:
The sequence is bounded, above by 1 and below by 0. It is not monotonic. However, the sequence of its odd-index terms is

$$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots$$

and the sequence of its even index terms is

$$\frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \dots$$

Both of these sequences are montonically decreasing and bounded below, so both are convergent by the Monotone Convergence Theorem. Since our sequence comes from just alternating terms of these two convergent sequences, it must be convergent too.

Questions to think about - answers on Blackboard please!

1. Has Anja proved that the sequence is convergent? If not, why not?
2. Has Bertha proved that the sequence is convergent? If not, why not?
3. Is Collette right? Do Collette's comments prove that we cannot decide if the sequence is convergent or not?
4. Has Deirdre proved that the sequence is convergent? If not, why not?
5. Has Eva proved that the sequence is convergent? If not, why not?
6. As a reader, what questions and/or feedback would you have for each of the authors?
7. Rank (some or all of) the responses in your order of preference.