Chapter 3

Sequences, Series and Convergence

3.1 Introduction to sequences and series

Example 3.1.1. *Does it make sense to talk about the "number"*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots ?$$

What does the question "does it make sense" mean? What we are talking about is the sum of infinitely many specified positive numbers. We can't actually do the addition and calculate what this "number" is based on the definition above. But we can add up any finite collection of the given terms and get an answer for that. Does this sum "settle down" to some identifiable value if we keep adding more terms (whatever that means)? Does it keep growing and growing without bound? Are there ways of finding out? Why would we want to know?

The following experiment might give a slightly vague but hopefully convincing answer to some of these questions. Partially evaluating the sum above for various "initial segments" gives the following results.

- $1 + \frac{1}{4} = 1.25$
- $1 + \frac{1}{4} + \frac{1}{9} \approx 1.361111$
- $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.423611$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10)^2} \approx 1.549767$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(200)^2} \approx 1.639947$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10000)^2} \approx 1.644834$

This experiment goes as far as computing the first 100000 terms of the sum, and it appears that the values are not increasing without limit but "settling down" at around 1.6449. What is the significance of this?

$$\frac{\pi^2}{6} \approx 1.644934$$

The *series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges* to the number $\frac{\pi^2}{6}$ (we will have precise definitions for the italicized terms a bit later). This fact is remarkable - there is no obvious connection between π and squares of the form $\frac{1}{n^2}$; moreover all the terms in the series are rational but $\frac{\pi^2}{6}$ is certainly not. This example gives us in principle a way of calculating the digits of π or at least of π^2. (In practice there are similar but better ways, as the convergence in this example is very slow).

Example 3.1.2. *What about*

$$\sum_{i=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots?$$

Try experimenting with initial segments again :

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There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course - maybe there is convergence but it can't be seen until we take many more millions of terms into our calculation? How could we know that this *doesn't* converge to anything?

Example 3.1.3. *What about*

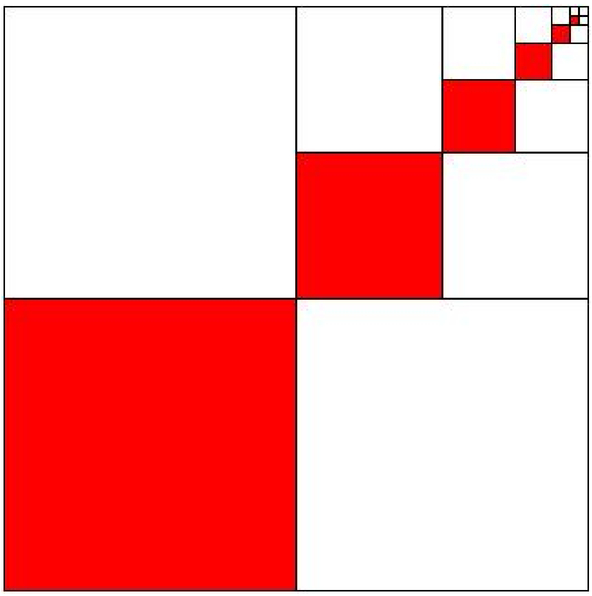
$$\sum_{i=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots?$$

Experimenting reveals

- $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$
- $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} = \frac{341}{1024} \approx 0.33301$

These calculations can be verified directly using properties of sums of geometric progressions. It appears that this series is converging to $\frac{1}{3}$.

The following picture gives some graphical evidence for this hypothesis. The large square has area 1, and the red squares have areas $\frac{1}{4}$, $\frac{1}{16}$, etc. The picture is intended to indicate that the red squares occupy one-third of the total area, since every red square is "accompanied" by two white squares of the same area, and all these squares together make up the total area 1. This picture is not really a proof, as it is not possible to actually draw squares representing all the terms of the series, but it is a visual way of understanding the statement that the series $\sum_{n=1}^{\infty} \frac{1}{2^{2n}}$ converges to $\frac{1}{3}$.



3.2 Sequences

Note: Chapter 11 of Stewart's Calculus is a good reference for this chapter of our lecture notes.

Definition 3.2.1. *A* sequence *is basically an infinite ordered list*

$$a_1, a_2, a_3, \dots$$

- The items in list a_1, a_2 etc. are called *terms* (1st term, 2nd term, and so on).
- In our context the terms will generally be real numbers - but they don't have to be.
- The sequence $a_1, a_2, \dots$ can be denoted by $\{a_n\}$ or by $\{a_n\}_{n=1}^{\infty}$.
- There may be an overall formula for the terms of the sequence, or a "rule" for getting from one to the next, but there doesn't have to be.

Example 3.2.2. *1.* $\{(-1)^n + 1\}_{n=1}^{\infty} : a_n = (-1)^n + 1$
$a_1 = -1 + 1 = 0,\ a_2 = (-1)^2 + 1 = 2, a_3 = (-1)^3 + 1 = 0, \dots$

$$0, 2, 0, 2, 0, 2, \dots$$

2. $\{\sin(\frac{n\pi}{2})\}_{n=1}^{\infty} : a_n = \sin(\frac{n\pi}{2})$
$a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \sin(\pi) = 0,\ a_3 = \sin(\frac{3\pi}{2}) = -1,\ a_4 = \sin(2\pi) = 0, \dots.$

$$1, 0, -1, 0, 1, 0, -1, 0, \dots$$

3. $\{\frac{1}{n}\sin(\frac{n\pi}{2})\}_{n=1}^{\infty} : a_n = \sin(\frac{n\pi}{2})$
$a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \frac{1}{2}\sin(\pi) = 0,\ a_3 = \frac{1}{3}\sin(\frac{3\pi}{2}) = -\frac{1}{3},\ a_4 = \frac{1}{4}\sin(2\pi) = 0, \dots.$

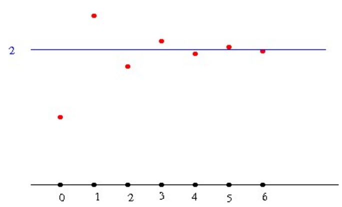
$$1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, 0, \dots$$

One way of visualizing a sequence is to consider is as a function whose domain is the set of natural numbers and think of its graph, which will be a collection of isolated points, one for each natural number.

Example 3.2.3. $\{2 + (-1)^n 2^{1-n}\}_{n=1}^{\infty}$. *Write* $a_n = 2 + (-1)^n 2^{1-n}$. *Then*

$$a_1 = 2 - 2^0 = 1,\ a_2 = 2 + 2^{-1} = \frac{5}{2},\ a_3 = 2 - 2^{-2} = \frac{7}{4}, a_4 = 2 + 2^{-3} = \frac{17}{8}.$$

Graphical representation of $\{a_n\}$*:*



As n gets very large the positive number $\frac{1}{2^{n-1}}$ gets very small. By taking n as large as we like, we can make $\frac{1}{2^{n-1}}$ as small as we like.

Hence for very large values of n, the number $2 + (-1)^n \frac{1}{2^{n-1}}$ is very close to 2. By taking n as large as we like, we can make this number as close to 2 as we like.

We say that the sequence *converges* to 2, or that 2 is the *limit* of the sequence, and write

$$\lim_{n\to\infty} \left(2 + (-1)^n \frac{1}{2^{n-1}}\right) = 2.$$

Note: Because $(-1)^n$ is alternately positive and negative as n runs through the natural numbers, the terms of this sequence are alternately greater than and less than 2.

We now state the formal definition of convergence of a sequence. This is reminiscent of the definition of a limit for a function. A sequence converges to the number L if no matter how restrictive your notion of "near L" is, there is a point in the sequence beyond which every term is near L.

Definition 3.2.4. *The sequence* $\{a_n\}$ converges *to the number* L *(or has* limit L*) if for every positive real number* ε *(no matter how small) there exists a natural number* N *with the property that the term* a_n *of the sequence is within* ε *of* L *for all terms* a_n *beyond the* N*th term. In more compact language :*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ for which } |a_n - L| < \varepsilon\ \forall n > N.$$

Notes

- If a sequence has a limit we say that it *converges* or *is convergent*. If not we say that it *diverges* or *is divergent*.
- If a sequence converges to L, it means that no matter how small a radius around L we choose, there is a point in the sequence beyond which all terms are within that radius of L. This means (at least) that beyond a certain point all terms of the sequence are *very close together* (and very close to L). Where that point is depends on how you interpret "very close together".

Being convergent is a very strong property for a sequence to have, and there are lots of different ways for a sequence to be divergent.

Example 3.2.5. *1.* $\{\max(\{(-1)^n, 0\})\}_{n=1}^{\infty}$: 0, 1, 0, 1, 0, 1, ...
This sequence alternates between 0 and 1 and does not approach any limit.

2. *A sequence can be divergent by having terms that increase (or decrease) without limit.*
$\{2^n\}_{n=1}^{\infty}$: 2, 4, 8, 16, 32, 64, ...
3. *A sequence can have haphazard terms that follow no overall pattern, such as the sequence whose* n*th term is the* n*th digit after the decimal point in the decimal representation of* π.

Remark: The notion of a convergent is sometimes described informally with words like "the terms get closer and closer to L as n gets larger". It is *not true* however that the terms in a sequence that converges to a limit L *must* get *progressively* closer to L as n increases, as the following example shows.

Example 3.2.6. *The sequence* a_n *is defined by*

$$a_n = 0 \textit{ if } n \textit{ is even, } a_n = \tfrac{1}{n} \textit{ if } n \textit{ is odd.}$$

This sequence begins :

$$1,\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{5},\ 0,\ \frac{1}{7},\ 0,\ \frac{1}{9},\ 0, \dots$$

It converges to 0 *although it is not true that every step takes us closer to zero.*

The following is an example of a convergent sequence.

Example 3.2.7. *Find* $\lim_{n\to\infty} \dfrac{n}{2n-1}$.
Solution*: As if calculating a limit as* $x \to \infty$ *of an expression involving a continuous variable* x*, divide above and below by* n.

$$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{n/n}{2n/n - 1/n} = \lim_{n\to\infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}.$$

So the sequence $\left\{\dfrac{n}{2n-1}\right\}$ *converges to* $\frac{1}{2}$.

As for subsets of $\mathbb{R}$, there is a concept of *boundedness* for sequences. Basically a sequence is bounded (or bounded above or bounded below) if the set of its terms, considered as a subset of $\mathbb{R}$, is bounded (or bounded above or bounded below). More precisely :

Definition 3.2.8. *The sequence* $\{a_n\}$ *is* bounded above *if there exists a real number* M *for which* $a_n \leqslant M$ *for all* $n \in \mathbb{N}$.
The sequence $\{a_n\}$ *is* bounded below *if there exists a real number* m *for which* $m \leqslant a_n$ *for all* $n \in \mathbb{N}$.
The sequence $\{a_n\}$ *is* bounded *if it is bounded both above and below.*

Example 3.2.9. *The sequence* $\{n\}$ *is bounded below (for example by* 0 *or* 1*) but not above. The sequence* $\{\sin n\}$ *is bounded below (for example by* -1*) and above (for example by* 1*).*

Theorem 3.2.10. *If a sequence is convergent it must be bounded.*

Proof

Note : what we have to do here is use the definitions of convergent and bounded to reason that every sequence that is convergent must be bounded.

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence with limit L.

Then (by definition of convergence) there exists a natural number N such that every term of the sequence after a_N is between $L-1$ and $L+1$.

(Note: there is nothing special here about $L-1$ *and* $L+1$ *- you could choose* $L-\frac{1}{2}$ *and* $L+\frac{1}{2}$ *or anything like that - the point is that when you choose a certain "window" around* L*, there is a point (*N*) beyond which all the terms of the sequence are in this "window".)*

The set consisting of the first N terms of the sequence is a finite set : it has a maximum element M_1 and a minimum element m_1.
Let $M = \max\{M_1, L+1\}$ and let $m = \min\{m_1, L-1\}$.

(So M *is defined to be either* M_1 *or* $L+1$*, whichever is the greater, and* m *is defined to be either* m_1 *or* $L-1$*, whichever is the lesser.)*

Then $\{a_n\}$ is bounded above by M and bounded below by m.
So our sequence is bounded.

INCREASING AND DECREASING SEQUENCES

Definition 3.2.11. *A sequence* $\{a_n\}$ *is called* increasing *if* $a_n \leqslant a_{n+1}$ *for all* $n \geqslant 1$.
A sequence $\{a_n\}$ *is called* strictly increasing *if* $a_n < a_{n+1}$ *for all* $n \geqslant 1$.
A sequence $\{a_n\}$ *is called* decreasing *if* $a_n \geqslant a_{n+1}$ *for all* $n \geqslant 1$.
A sequence $\{a_n\}$ *is called* strictly decreasing *if* $a_n > a_{n+1}$ *for all* $n \geqslant 1$.

Definition 3.2.12. *A sequence is called* monotonic *if it is* either increasing or decreasing.
Similar terms : monotonic increasing, monotonic decreasing, monotonically increasing/decreasing.

Note: These definitions are not *entirely* standard. Some authors use the term *increasing* for what we have called *strictly increasing* and/or use the term *nondecreasing* for what we have called *increasing*.
Examples

1. An increasing sequence is bounded below but need not be bounded above. For example

$$\{n\}_{n=1}^{\infty} : 1,\ 2,\ 3,\ \dots$$

2. A bounded sequence need not be monotonic. For example

$$\{(-1)^n\} : -1,\ 1,\ -1,\ 1,\ -1,\ \dots$$

3. A convergent sequence need not be monotonic. For example
$$\left\{\frac{(-1)^{n+1}}{n}\right\}_{n=1}^{\infty} : 1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \dots$$
This sequence converges to 0 but is neither increasing nor decreasing.
4. A montonic sequence need not be convergent, as Example 1 above shows.

However, if a sequence is *bounded* and *monotonic*, it is *convergent*. This is the *Monotone Convergence Theorem*, which is the major theorem of this section.

Theorem 3.2.13 (The Monotone Convergence Theorem)**.** *If a sequence* $\{a_n\}_{n=1}^{\infty}$ *is montonic and bounded, then it is convergent.*

Proof: *(We can start by giving ourselves a monotonic bounded sequence - we can take it to be increasing; the argument for a decreasing sequence is similar.)*
Suppose that $\{a_n\}$ is *increasing* and bounded. Then the set $\{a_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$ and by the *Axiom of Completeness* it has a *least upper bound* (or supremum) L.

(We are just giving the name L *here to the supremum of the set of values of the sequence. We are supposed to be showing that the sequence is convergent, i.e. has a limit :* L *is our candidate for that limit)*

We will show that the sequence $\{a_n\}$ converges to L.

Choose a (very small) $\varepsilon > 0$. Then $L-\varepsilon$ is *not an upper bound* for $\{a_n : n \in \mathbb{N}\}$, becasue L is the *least* upper bound for this set.
This means there is some $N \in \mathbb{N}$ for which $L - \varepsilon < a_N$. Since L is an upper bound for $\{a_n : n \in \mathbb{N}\}$, this means
$$L - \varepsilon < a_N \leqslant L$$
(i.e. a_N *is between* $L-\varepsilon$ *and* L*).*

Since the sequence $\{a_n\}$ is increasing and its terms are bounded above by L, *every* term after a_N is between a_N and L, and therefore between $L-\varepsilon$ and L. These terms are all within ε of L.
Using the fact that our sequence is increasing and bounded, we have

- Identified L as the least upper bound for the set of terms in our sequence
- Showed that no matter how small an ε we take, there is a point in our sequence beyond which *all* terms are within ε of L.

This is exactly what it means for the sequence to converge to L. This concludes the proof.

Example 3.2.14 (from 2011 Summer Exam)**.** *A sequence* $\{a_n\}$ *is defined by*
$$a_1 = 0, \quad a_{n+1} = \sqrt{a_n + 6} \text{ for all } n \geqslant 1.$$
Show that this sequence is bounded above by 3 and that it is increasing.
Deduce that the sequence is convergent and find its limit.

Note: This is an example of a sequence that is defined *recursively*. This means that the first term is given and subsequent terms are defined (one by one) in terms of previous ones. We are not given a general formula for the nth term although one may exist.

Solution:

1. 3 *is an upper bound.*
Suppose that $a_k < 3$ for some k. Then
$$a_{k+1} = \sqrt{a_k + 6} < \sqrt{3+6} = 3.$$
This says that if $a_k \leqslant 3$, then $a_{k+1} \leqslant 3$ also.
Then, since $a_1 < 3$, we have $a_2 < 3$, then $a_3 < 3$, etc.

2. *The sequence is increasing*
   Let $k \in \mathbb{N}$. We need to show that $a_k \leqslant a_{k+1}$.
   We know that $0 \leqslant a_k < 3$: note this implies that

   $$a_k = \sqrt{a_k^2} < \sqrt{3a_k} = \sqrt{a_k + 2a_k} < \sqrt{a_k + 6} = a_{k+1}.$$

   Then $a_k < a_{k+1}$ for each k, which means the sequence is increasing.

3. *The sequence converges*
   Since the sequence is increasing and bounded, it converges by the Monotone Convergence Theorem.

   Let L be the limit. Then, taking limits as $n \to \infty$ on both sides of the equation

   $$a_{n+1} = \sqrt{a_n + 6}$$

   we find that

   $$L = \sqrt{L+6} \Longrightarrow L^2 = L + 6 \Longrightarrow (L-3)(L+2) = 0.$$

   Thus $L = 3$ or $L = -2$, and since all the terms of our sequence are between 0 and 3 it must be that $L = 3$.

3.3 Introduction to Infinite Series

Definition 3.3.1. *A* series *or* infinite *series is the sum of all the terms in a sequence.*

Example 3.3.2 (Examples of infinite series)**.** *1.* $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$

2. A geometric *series*

$$\sum_{n=1}^{\infty} \frac{1/2^n}{=} 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

Every term in this series is obtained from the previous one by multiplying by the common ratio $\frac{1}{2}$. *This is what* geometric *means.*

3. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

4. An alternating *series*

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

Notes

1. For now these infinite sums are just formal expressions or arrangements of symbols. Whether it is meaningful to think of them as numbers or not is something that can be investigated.

2. A *series* is not the same thing as a *sequence* and it is important not to confuse these terms. A *sequence* is just a list of numbers. A *series* is an infinite sum.

3. The "sigma" notation for sums : *sigma* (lower case σ, upper case Σ) is a letter from the Greek alphabet, the upper case $\sum$ is used to denote sums. The notation

$$\sum_{n=i}^{j} a_n$$

means : i and j are integers and $i \leqslant j$. For each n from i to j the number a_n is defined; the expression above means the sum of the numbers a_n where n runs through all the values from i to j, i.e.

$$\sum_{n=i}^{j} a_n = a_i + a_{i+1} + a_{i+2} + \dots + a_{j-1} + a_j.$$

For example

$$\sum_{n=2}^{5} n^2 = 2^2 + 3^2 + 4^2 + 5^2 = 54.$$

For infinite sums it is possible to have $-\infty$ and/or ∞ (instead of fixed integers i and j) as subscripts and superscripts for the summation.

What does it mean to talk about the sum of *infinitely many* numbers? We cannot add infinitely many numbers together in practice, although we can (in principle) at least, add up any finite collection of numbers. In the examples above we can start from the beginning, adding terms at the start of the series. Adding term by term we get the following lists.

1. $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$

$$1, 1+2, 1+2+3, 1+2+3+4, 1+2+3+4+5, \dots : 1, 3, 6, 10, 15, \dots$$

Since the terms being added on at each stage are getting bigger, the numbers in the list above will keep growing (faster and faster as n increases) - we can't associate a numberical value with this infinite sum.

2. A *geometric* series

$$\sum_{n=1}^{\infty} \frac{1/2^n}{=} 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\dots \;:\; 1,\ \frac{3}{2},\ \mathrm{frac74},\ \mathrm{frac158},\ \frac{31}{16},\ \frac{63}{32}\dots$$

In this example the terms that are being added on at each step ($\frac{1}{2^n}$) are getting smaller and smaller as n increases, and the numbers in the list appear to be converging to 2.

3. The *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\dots \;:\; 1,\ \frac{3}{2},\ \frac{11}{6},\ \frac{25}{12},\ \frac{137}{60},\dots$$

It is harder to see what is going on here.

4. An *alternating* series

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

$$1,\ 1-1,\ 1-1+1,\ 1-1+1-1,\ 1-1+1-1+1\dots \;:\; 1, 0, 1, 0, 1, \dots$$

The terms being "added on" at each step are alternating between 1 and -1, and as we proceed with the summation the "running total" alternates between 0 and 1. So there is no numerical value that we can associate with the infinite sum $\sum_{n=0}^{\infty}(-1)^n$.

Note: The series in 2. above *converges* to 2, the series in 1. and 4. are both *divergent* and it is not obvious yet but the series in 3. is *divergent* as well. Our next task is to give precise meanings to these terms for series. In order to do this we need some terminology. Bear in mind that we know what it means for a sequence to converge, but we don't yet have a definition of convergence for series.

Definition 3.3.3. *For a series* $\sum_{n=1}^{\infty} a_n$*, and for* $k \geqslant 1$*, let*

$$s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \dots + a_k.$$

Thus $s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3$ *etc.*
Then s_k *is called the* kth partial sum *of the series, and the sequence* $\{s_k\}_{k=1}^{\infty}$ *is called the* sequence of partial sums *of the series.*
If the sequence of partial sums converges to a limit s*, the series is said to* converge *and* s *is called its sum. In this situation we can write*

$$\sum_{n=1}^{\infty} a_n = s.$$

If the sequence of partial sums diverges*, the series is said to* diverge.

Example 3.3.4 (Convergence of a geometric series). *Recall the second example above :*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

In this example, for $k \geqslant 0$,

$$\begin{aligned} s_k = \sum_{n=0}^{k} \frac{1}{2^n} = 1 \quad &+ \quad \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2^k} \\ \frac{1}{2} s_k = \sum_{n=1}^{k} \frac{1}{2^{n+1}} = \quad & \quad \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2^k} + \frac{1}{2^{k+1}} \end{aligned}$$

Then

$$s_k - \frac{1}{2} s_k = \frac{1}{2} s_k = 1 - \frac{1}{2^{k+1}} \Longrightarrow s_k = 2 - \frac{1}{2^k}.$$

So the sequence of partial sums has kth term $2 - \frac{1}{2^k}$. This sequence converges to 2 so the series converges to 2; we can write

$$\sum_{n} = 0^{\infty} \frac{1}{2^k} = 2.$$

General geometric series : Consider the sequence of partial sums for the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$$

(This is a *geometric series* with initial term a and *common ratio* r.) The kth partial sum s_k is given by

$$\begin{aligned} s_k = \sum_{n=0}^{k} ar^n = a \quad &+ \quad ar + \dots ar^k \\ rs_k = \sum_{n=0}^{k} ar^{n+1} = \quad & \quad ar + ar^2 + \dots ar^{k+1} \end{aligned}$$

Then $(1 - r)s_k = a - ar^{k+1} \Longrightarrow s_k = \dfrac{a(1 - r^{k+1})}{1 - r}$. If $|r| < 1$, then $r^{k+1} \to 0$ as $k \to \infty$, and the sequence of partial sums (hence the series) converges to $\dfrac{a}{1 - r}$. If $|r| \geqslant 1$ the series is divergent.

Next we show that the harmonic series is divergent.

Theorem 3.3.5. *The harmonic series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

Proof: We show that the sequence of partial sums of the harmonic series is not bounded above.

- The first term is 1.
- The second term is $\frac{1}{2}$.
- The sum of the 3rd and 4th terms exceeds $\frac{1}{2}$:

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

- The sum of the 5th, 6th, 7th and 8th terms exceeds $\frac{1}{2}$:

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}.$$

- For the same reason, the sum of the next 8 terms (terms 9 through 16) also exceeds $\frac{1}{2}$.

- In general the sum of the 2^{n-1} terms $\dfrac{1}{2^{n-1}+1}$ through $\dfrac{1}{2^n}$ exceeds $\frac{1}{2}$.

So, as we list terms in the sequence of partial sums of the harmonic series, we keep accumulating non-overlapping stretches of terms that add up to more than $\frac{1}{2}$. Thus the entire series has infinitely many non-overlapping stretches all individually summing to more than $\frac{1}{2}$. Then the sum of this series is not finite and the series diverges.

Note: A necessary condition for the series $\sum_{n=1}^{\infty} a_n$ to *converge* is that the *sequence* $\{a_n\}_{n=1}^{\infty}$ converges to 0; i.e. that $a_n \to 0$ as $n \to \infty$. If this does *not* happen, then the sequence of partial sums has no possibility of converging.

The example of the harmonic series shows that the condition $a_n \to 0$ as $n \to \infty$ is not *sufficient* to guarantee that the series $\sum_{n=1}^{\infty}$ will converge.

3.4 Introduction to power series

Definition 3.4.1. *A* polynomial *in the* variable x *is an expression of the form*

$$\sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1}x^{n-1} + a_n x^n$$
$$\text{or} \quad a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0,$$

where the coefficients a_i *are real numbers and* n *is a natural number. The* degree *of the polynomial is the highest* k *for which* x^k *appears with non-zero coefficient.*

Examples

1. $x^3 + 5x^2$ is a polynomial of degree 3 (also called cubic).
2. $2 + 2x + 2x^2 + 2x^7$ is a polynomial of degree 7.
3. $x^4 + x^3 + \frac{3}{x^2}$ is *not* a polynomial, because it involves a negative power of x.

The point is that a polynomial can has a constant term (which may be zero) and a *finite number* of terms involving particular positive powers of x that have numbers as coefficients. A polynomial may be regarded as a function of x, and polynomials are functions of a special type.

Definition 3.4.2. *A* power series *in the variable* x *resembles a polynomial, except that it may contain infinitely many positive powers of* x. *It is an expression of the type*

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots,$$

where each a_i *is a number.*

Example 3.4.3.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

is a power series.

Question: Does it make sense to think of a power series as a *function* of x? We investigate this question for the example above.

So define a "function" by

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

- If we try to evaluate this function at $x = 2$, we get a *series* of real numbers.

  $$f(2) = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \dots$$

  This series is divergent, so our power series does not define a function that can be evaluated at 2.

- If we try evaluating at 0 (and allow that the first term x^0 of the power series is interpreted as 1 for *all* values of x), we get

  $$f(0) = 1 + 0 + 0^2 + \dots = 1.$$

  So it does make sense to "evaluate" this function at $x = 0$.

- If we try evaluating at $x = \frac{1}{2}$, we get

$$f(\frac{1}{2}) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots$$

  From our work on geometric series in Section 2.3 we know that this is a geometric series with first term $a = 1$ and common ratio $r = \frac{1}{2}$. We know that if $|r| < 1$, such a series converges to the number $\dfrac{a}{1-r}$. In this case

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2,$$

  and we have $f(\frac{1}{2}) = 2$.

In general we know that a geometric series of this sort converges provided that the absolute value of its common ratio is less than 1. So for example if we put $x = \frac{1}{3}$ we find that $f(\frac{1}{3})$ is the sum of a geomtric series with first term 1 and common ratio $\frac{1}{3}$; this is

$$\frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

In general for any value of x whose absolute value is less than 1 (i.e. any x in the interval $(-1,1)$), we find that $f(x)$ is a convergent geometric series, converging to $\dfrac{1}{1-x}$.

Conclusion: For values of x in the interval $(-1,1)$ (i.e. $|x| < 1$), the function $f(x) = \frac{1}{1-x}$ coincides with the power series $\sum_{n=0}^{\infty} x^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

The interval $(-1,1)$ is called the *interval of convergence* of the power series, and 1 is the *radius of convergence*. We say that the *power series representation* of the function $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$, for values of x in the interval $(-1,1)$.
Note: The expression $\frac{1}{1-x}$ makes sense of course for all values of x except $x = 1$. We are not saying that the domain of the function $f(x) = \frac{1}{1-x}$ only consists of the interval $(-1,1)$, but just that it is only on this interval that our power series represents this function.

Remark: The fact that for certain values of x we can represent $\frac{1}{1-x}$ with a power series might be interesting (at least to some people!), but it is not of particular use if you want to calculate $\frac{1}{1-x}$ for some particular value of x, because this is easily done directly. However, if we could obtain a power series representation for a function like $\sin x$ and use it to evaluate (or approximate) $sin(1)$ or $\sin(9)$ or $\sin(20)$, that might be of real practical use. These numbers are *not* easy to obtain directly because the definition of $\sin x$ doesn't tell us how to calculate $\sin x$ for a particular x - you can use a calculator of course but how does the calculator do it? If we had a power series representation for $\sin x$ and we knew it converged for the value of x we had in mind, we couldn't necessarily write down the limit but we could calculate partial sums to get an estimation as accurate as we like.

Questions: What functions can be represented by power series, and on what sorts of interval or subsets of $\mathbb{R}$? If a function could be represented by a power series, how would we calculate the coefficients in this series?

We are not going to give a full answer to these questions, but a partial one involving *Maclaurin* or *Taylor* series.

Suppose that $f(x)$ is an infinitely differentiable function (this means that all the deriviatives of f are themselves differentiable), and suppose that f is represented by the power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We can work out appropriate values for the coefficients c_n as follows.

- Put $x = 0$. Then $f(0) = c_0 + \sum_{n=1}^{\infty} c_n (0)^n \implies f(0) = c_0$.
  The constant term in the power series is the value of f at 0.

- To calculate c_1, look at the value of the first derivative of f at 0, and differentiate the power series term by term. We expect

  $$f'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots = \sum_{n=1}^{\infty} nc_n x^{n-1}.$$

  Then we should have

  $$f'(0) = c_1 + 2c_2 \times 0 + 3c_3 \times 0 + \cdots = c_1.$$

  Thus $c_1 = f'(0)$.

- For c_2, look at the second derivative of f. We expect

  $$f''(x) = 2(1)c_2 + 3(2)c_3x + 4(3)c_4x^2 + 5(4)c_5x^3 + \cdots = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

  Putting $x = 0$ gives $f''(0) = 2(1)c_2$ or $c_2 = \dfrac{f''(0)}{2(1)}$.

- For c_3, look at the third derivative $f^{(3)}(x)$. We have

  $$f^{(3)}(x) = 3(2)(1)c_3 + 4(3)(2)c_4x + 5(4)(3)c_5x^2 + \cdots = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n x^{n-3}.$$

  Setting $x = 0$ gives $f^{(3)}(0) = 3(2)(1)c_3$ or $c_3 = \frac{f^{(3)}(0)}{3(2)(1)}$

Continuing this process, we obtain the following general formula for c_n:

$$c_n = \frac{1}{n!} f^{(n)}(0).$$

Definition 3.4.4. *For a positive integer* n, *the number* n factorial, *denoted* $n!$ *is defined by*

$$n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1.$$

The number $0!$ *(zero factorial) is defined to be 1.*

Example 3.4.5 (Power series representation of e^x)**.**

The coefficient of x^n in the Maclaurin series expansion of e^x is

$$c_n = \frac{1}{n!} \frac{d^n}{dx^n} (e^x)|_{x=0} = \frac{1}{n!} e^0 = \frac{1}{n!}.$$

Thus the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Note that if we differentiate this series term by term we get exactly the same series back, which is what we would expect for a power series that represents e^x, since $\dfrac{d}{dx}(e^x) = e^x$.

Theorem 3.4.6. *For* every *real number* x*, the above power series converges to* e^x*. The interval of convergence of this power series is all of* $\mathbb{R}$ *and the radius of convergence is infinite.*

Example 3.4.7 (Power series representation of $\sin x$)**.**

Write $f(x) = \sin x$, and write $\sum_{n=0}^{\infty} c_n x^n$ for the Maclaurin series of $\sin x$. Then

- $f(0) = \sin 0 = 0 \Longrightarrow c_0 = 0$
- $f'(0) = \cos 0 = 1 \Longrightarrow c_1 = 1$
- $f''(0) = -\sin 0 = 0 \Longrightarrow c_2 = \frac{0}{2!} = 0$
- $f^{(3)}(0) = -\cos 0 = -1 \Longrightarrow c_3 = \frac{-1}{3!} = -\frac{1}{6}$
- $f^{(4)}(0) = \sin 0 = 0 \Longrightarrow c_4 = \frac{0}{4!} = 0$

This pattern continues : if k is even then $f^{(k)}(0) = \pm sin 0 = 0$, so $c_k = 0$.
If k is odd and $k \equiv 1 \mod 4$ then $f^{(k)}(0) = \cos 0 = 1$ and $c_k = \frac{1}{k!}$.
If k is odd and $k \equiv 3 \mod 4$ then $f^{(k)}(0) = -\cos 0 = -1$ and $c_k = -\frac{1}{k!}$.
Thus the Maclaurin series for $\sin x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

Note that this series only involves odd powers of x - this is not surprising because sin is an *odd function*; it satisfies $\sin(-x) = -\sin x$.

Theorem 3.4.8. *For every real number* x*, the above series converges to* $\sin x$*.*

Thus computing partial sums of this series gives us an effective way of approximating $\sin x$ for any real number x.

Exercise 3.4.9. *Show that the Maclaurin series for* $\cos x$ *is given by*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k!)} x^{2k}.$$

(Note that this can be obtained by differentiating term-by-term the series for $\sin x$, as we would expect since $\frac{d}{dx}(\sin x) = \cos x$.)

3.5 Exam advice and sample question for Chapter 3

Here is a "sample Question 3". For this question you will need to be able to use the concept of convergence and demonstrate a clear understanding of what it means. You will need to be able to state and apply the main theoretical elements of Chapter 3, such as the Monotone Convergence Theorem and the statement that every convergent sequence is bounded.

3. (a) Give an example of
    - i. a convergent sequence of real numbers;
    - ii. a sequence of real numbers that is bounded and divergent;
    - iii. a sequence of real numbers that is strictly monotonically increasing;
    - iv. a sequence of real numbers that is convergent and is not monotonic.

   (b) A sequence (a_n) of real numbers is defined by

$$a_0 = 4, \ a_n = \sqrt{a_{n-1}^2 - 2a_{n-1} + 4} \text{ for } n \geqslant 1.$$

i. Write down the first four terms of the sequence.
ii. Show that the sequence is bounded below by 2.
iii. Show that the sequence is montonically decreasing.
iv. State why it can be deduced that the sequence is convergent, and determine its limit.

(c) Find the first four terms in the Maclaurin series of $\dfrac{1}{1-x}$.

Sample Solution:

(a) i (a_n) defined by $a_n = \frac{1}{n}$ for $n \geqslant 1$.
ii (a_n) defined by $a_n = (-1)^n$, for $n \geqslant 1$.
iii (a_n) defined by $a_n = n,\ n \geqslant 1$.
iv (an) defined by $a_n = (-1)^n \frac{1}{n}$, for $n \geqslant 1$.

(b) i $a_0 = 4,\ a_1 = \sqrt{12},\ a_2 = \sqrt{16 - 2\sqrt{12}},\ a_3 = \sqrt{16 - 2\sqrt{12} - 2\sqrt{16 - 2\sqrt{12}} + 4}$
ii Certainly $a_0 > 2$. Suppose that $a_k > 2$ for some k. Then $a_k^2 - 2a_k > 0$ and

$$a_{k+1} = \sqrt{a_k^2 - 2a_k + 4} > \sqrt{4} \Longrightarrow a_{k+1} > 2.$$

iii We know that $a_k > 2$ for $k \in \mathbb{N}$. Then $4 - 2a_k < 0$ and

$$a_{k+1} = \sqrt{a_k^2 - 2a_k + 4} < \sqrt{a_k^2} \Longrightarrow a_{k+1} < a_k.$$

iv Since the sequence (a_n) is bounded below and monotonically decreasing, it is convergent by the Monotone Convergence Theorem. Its limit L must satisfy

$$L = \sqrt{L^2 - 2L + 4} \Longrightarrow L^2 = L^2 - 2L + 4 \Longrightarrow 2L = 4 \Longrightarrow L = 2.$$

(c) Write $f(x) = \dfrac{1}{1-x}$. Then

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \\ \frac{1}{2}f''(0) &= \frac{1}{2}(2) = 1 \\ \frac{1}{3}f^{(3)}(0) &= \frac{1}{6}(6) = 1 \end{aligned}$$

First four terms of Maclaurin series: $1,\ x,\ x^2, x^3$.