

Section 3.4 : Introduction to power series

Definition 89

A **polynomial** in the **variable** x is an expression of the form

$$\sum_{i=0}^n a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

or $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$

where the coefficients a_i are real numbers and n is a natural number. The **degree** of the polynomial is the highest k for which x^k appears with non-zero coefficient.

Examples

- 1 $x^3 + 5x^2$ is a polynomial of degree 3 (also called cubic).
- 2 $2 + 2x + 2x^2 + 2x^7$ is a polynomial of degree 7.
- 3 $x^4 + x^3 + \frac{3}{x^2}$ is **not** a polynomial, because it involves a negative power of x .

Power Series

Definition 90

A **power series** in the variable x resembles a polynomial, except that it may contain **infinitely many** positive powers of x . It is an expression of the type

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots,$$

where each a_i is a number.

Example 91

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

is a power series.

Question: Does it make sense to think of a power series as a **function** of x ?

Power Series as Functions

Define a “function” by

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

- If we try to evaluate this function at $x = 2$, we get a **series** of real numbers.

$$f(2) = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \dots$$

This series is divergent, so our power series does not define a function that can be evaluated at 2.

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- If we try evaluating at 0 (and allow that the first term x^0 of the power series is interpreted as 1 for *all* values of x), we get

$$f(0) = 1 + 0 + 0^2 + \dots = 1.$$

So it does make sense to “evaluate” this function at $x = 0$.

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

- If we try evaluating at $x = \frac{1}{2}$, we get

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots$$

This is a geometric series with first term $a = 1$ and common ratio $r = \frac{1}{2}$. We know that if $|r| < 1$, such a series converges to the number $\frac{a}{1-r}$. In this case

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2,$$

and we have $f\left(\frac{1}{2}\right) = 2$.

So we **can** evaluate our function at $x = \frac{1}{2}$.

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ for } |x| < 1$$

A geometric series of this sort converges provided that the absolute value of its common ratio is less than 1. In general for any value of x whose absolute value is less than 1 (i.e. any x in the interval $(-1, 1)$), we find that $f(x)$ is a convergent geometric series, converging to $\frac{1}{1-x}$.

Conclusion: For values of x in the interval $(-1, 1)$ (i.e. $|x| < 1$), the function $f(x) = \frac{1}{1-x}$ coincides with the power series $\sum_{n=0}^{\infty} x^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ for } |x| < 1.$$

The interval $(-1, 1)$ is called the **interval of convergence** of the power series, and 1 is the **radius of convergence**. We say that the **power series representation** of the function $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$, for values of x in the interval $(-1, 1)$.

Which functions have power series representations?

Remark: The power series representation is not particularly useful if you want to calculate $\frac{1}{1-x}$ for some particular value of x , because this is easily done directly. However, if we could obtain a power series representation for a function like $\sin x$ and use it to evaluate (or approximate) $\sin(1)$ or $\sin(9)$ or $\sin(20)$, that might be of real practical use. These numbers are **not** easy to obtain directly because the definition of $\sin x$ doesn't tell us how to calculate $\sin x$ for a particular x - you can use a calculator of course but how does the calculator do it?

Questions: What functions can be represented by power series, and on what sorts of interval or subsets of \mathbb{R} ? If a function could be represented by a power series, **how would we calculate the coefficients in this series?**

Maclaurin (or Taylor) series

Suppose that $f(x)$ is an infinitely differentiable function (this means that all the derivatives of f are themselves differentiable), and suppose that f is represented by the power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We can work out appropriate values for the coefficients c_n as follows.

- Put $x = 0$. Then $f(0) = c_0 + \sum_{n=1}^{\infty} c_n (0)^n \implies f(0) = c_0$.

The constant term in the power series is the value of f at 0.

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The constant term in the power series is the value of f at 0.
- To calculate c_1 , look at the value of the **first derivative** of f at 0, and differentiate the power series term by term. We expect

$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + \dots = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

Then we should have $f'(0) = c_1 + 2c_2 \times 0 + 3c_3 \times 0 + \dots = c_1$. Thus

$$c_1 = f'(0).$$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

- For c_2 , look at the second derivative of f . We expect

$$f''(x) = 2(1)c_2 + 3(2)c_3x + 4(3)c_4x^2 + 5(4)c_5x^3 + \dots$$

Putting $x = 0$ gives $f''(0) = 2(1)c_2$ or

$$c_2 = \frac{f''(0)}{2(1)}.$$

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Putting $x = 0$ gives $f''(0) = 2(1)c_2$ or

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- For c_3 , look at the third derivative $f^{(3)}(x)$. We have

$$f^{(3)}(x) = 3(2)(1)c_3 + 4(3)(2)c_4x + 5(4)(3)c_5x^2 + \dots$$

Setting $x = 0$ gives $f^{(3)}(0) = 3(2)(1)c_3$ or

$$c_3 = \frac{f^{(3)}(0)}{3(2)(1)}$$

Coefficients of the Maclaurin Series

Continuing this process, we obtain the following general formula for c_n :

$$c_n = \frac{1}{n!} f^{(n)}(0).$$

Definition 92

For a positive integer n , the number n factorial, denoted $n!$ is defined by

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 3 \times 2 \times 1.$$

The number $0!$ (zero factorial) is defined to be 1.

Power series representation of e^x

Example 93

The coefficient of x^n in the Maclaurin series expansion of e^x is

$$c_n = \frac{1}{n!} \frac{d^n}{dx^n} (e^x) \Big|_{x=0} = \frac{1}{n!} e^0 = \frac{1}{n!}.$$

Thus the Maclaurin series for e^x is given by $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

Note that if we differentiate this series term by term we get exactly the same series back, which is what we would expect for a power series that represents e^x , since $\frac{d}{dx}(e^x) = e^x$.

Theorem 94

For every real number x , the above power series converges to e^x . The interval of convergence of this power series is all of \mathbb{R} and the radius of convergence is infinite.

Power series representation of $\sin x$

Write $f(x) = \sin x$, and write $\sum_{n=0}^{\infty} c_n x^n$ for the Maclaurin series of $\sin x$. Then

- $f(0) = \sin 0 = 0 \implies c_0 = 0$

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- $f'(0) = \cos 0 = 1 \implies c_1 = 1$

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- $f''(0) = -\sin 0 = 0 \implies c_2 = \frac{0}{2!} = 0$
- $f^{(3)}(0) = -\cos 0 = -1 \implies c_3 = \frac{-1}{3!} = -\frac{1}{6}$

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- $f''(0) = -\sin 0 = 0 \implies c_2 = \frac{0}{2!} = 0$
- $f^{(3)}(0) = -\cos 0 = -1 \implies c_3 = \frac{-1}{3!} = -\frac{1}{6}$
- $f^{(4)}(0) = \sin 0 = 0 \implies c_4 = \frac{0}{4!} = 0$

Power series representation of $\sin x$

This pattern continues :

- If k is even then $f^{(k)}(0) = \pm \sin 0 = 0$, so $c_k = 0$.

Thus the Maclaurin series for $\sin x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

Note that this series only involves odd powers of x - this is not surprising because \sin is an **odd function**; it satisfies $\sin(-x) = -\sin x$.

Power series representation of $\sin x$

This pattern continues :

- If k is even then $f^{(k)}(0) = \pm \sin 0 = 0$, so $c_k = 0$.
- If k is odd and $k \equiv 1 \pmod{4}$ then $f^{(k)}(0) = \cos 0 = 1$ and $c_k = \frac{1}{k!}$.

Thus the Maclaurin series for $\sin x$ is given by

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- If k is even then $f^{(k)}(0) = \pm \sin 0 = 0$, so $c_k = 0$.
- If k is odd and $k \equiv 1 \pmod{4}$ then $f^{(k)}(0) = \cos 0 = 1$ and $c_k = \frac{1}{k!}$.
- If k is odd and $k \equiv 3 \pmod{4}$ then $f^{(k)}(0) = -\cos 0 = -1$ and $c_k = -\frac{1}{k!}$.

Thus the Maclaurin series for $\sin x$ is given by

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Power series representations of $\sin x$ and $\cos x$

Theorem 95

For every real number x , the above series converges to $\sin x$.

Thus computing partial sums of this series gives us an effective way of approximating $\sin x$ for any real number x .

Exercise 96

Show that the Maclaurin series for $\cos x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k!)} x^{2k}.$$

(Note that this can be obtained by differentiating term-by-term the series for $\sin x$, as we would expect since $\frac{d}{dx}(\sin x) = \cos x$.)

Learning outcomes for Section 3.4

After studying this section you should be able to

- State the meaning of the term *power series*,
- Explain the concept of the *radius of convergence* of a power series,
- Calculate the coefficients in (an initial segment of) the Maclaurin series representation of a given function.