Section 3.2 : Sequences

Note: Chapter 11 of Stewart's Calculus is a good reference for this chapter of our lecture notes.

Definition 71

A sequence is an infinite ordered list

$$a_1, a_2, a_3, \ldots$$

- The items in list a_1, a_2 etc. are called terms (1st term, 2nd term, and so on).
- In our context the terms will generally be real numbers - but they don't have to be.
- The sequence $a_1, a_2, \ldots$ can be denoted by (a_n) or by $(a_n)_{n=1}^{\infty}$.
- There may be an overall formula for the terms of the sequence, or a "rule" for getting from one to the next, but there doesn't have to be.

A Few Examples

1. $((-1)^n + 1)_{n=1}^{\infty}$: $a_n = (-1)^n + 1$
   $a_1 = -1 + 1 = 0,\ a_2 = (-1)^2 + 1 = 2, a_3 = (-1)^3 + 1 = 0, \dots$

$$0, 2, 0, 2, 0, 2, \dots$$

A Few Examples

1. $((-1)^n + 1)_{n=1}^{\infty}$: $a_n = (-1)^n + 1$
   $a_1 = -1 + 1 = 0,\ a_2 = (-1)^2 + 1 = 2, a_3 = (-1)^3 + 1 = 0, \ldots$

   $$0, 2, 0, 2, 0, 2, \ldots$$

2. $(\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$: $a_n = \sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \sin(\pi) = 0,\ a_3 = \sin(\frac{3\pi}{2}) = -1,\ a_4 = \sin(2\pi) = 0, \ldots.$

   $$1, 0, -1, 0, 1, 0, -1, 0, \ldots$$

A Few Examples

1. $((-1)^n + 1)_{n=1}^{\infty} \; : \; a_n = (-1)^n + 1$
   $a_1 = -1 + 1 = 0, \;\; a_2 = (-1)^2 + 1 = 2, a_3 = (-1)^3 + 1 = 0, \ldots$

   $$0, 2, 0, 2, 0, 2, \ldots$$

2. $(\sin(\frac{n\pi}{2}))_{n=1}^{\infty} \; : \; a_n = \sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1, \;\; a_2 = \sin(\pi) = 0, \;\; a_3 = \sin(\frac{3\pi}{2}) = -1, \;\; a_4 = \sin(2\pi) = 0, \ldots.$

   $$1, 0, -1, 0, 1, 0, -1, 0, \ldots$$

3. $(\frac{1}{n}\sin(\frac{n\pi}{2}))_{n=1}^{\infty} \; : \; a_n = \sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1, \;\; a_2 = \frac{1}{2}\sin(\pi) = 0, \;\; a_3 = \frac{1}{3}\sin(\frac{3\pi}{2}) = -\frac{1}{3}, \;\; a_4 = \frac{1}{4}\sin(2\pi) = 0, \ldots.$

   $$1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, 0, \ldots$$

Visualising a sequence

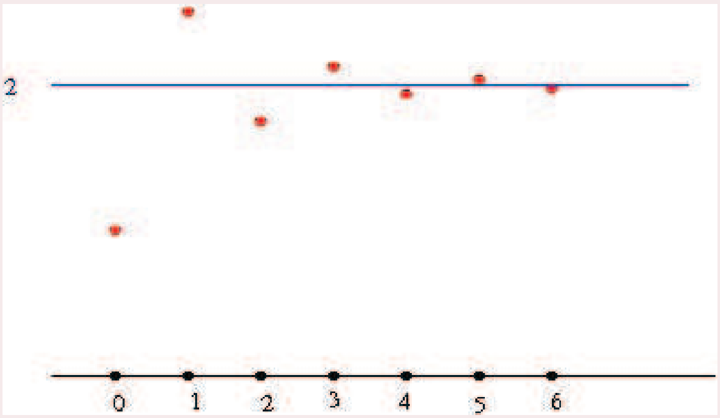
One way of visualizing a sequence is to consider is as a function whose domain is the set of natural numbers and think of its graph, which will be a collection of isolated points, one for each natural number.

Example 72

$(2+(-1)^n 2^{1-n})_{n=1}^{\infty}$. *Write* $a_n = 2+(-1)^n 2^{1-n}$. *Then*

$$a_1 = 2-2^0 = 1,\ a_2 = 2+2^{-1} = \frac{5}{2},\ a_3 = 2-2^{-2} = \frac{7}{4}, a_4 = 2+2^{-3} = \frac{17}{8}.$$

Graphical representation of (a_n)*:*



The sequence $(2 + (-1)^n \frac{1}{2^{n-1}})_{n=1}^{\infty}$

As n gets very large the positive number $\frac{1}{2^{n-1}}$ gets very small. By taking n as large as we like, we can make $\frac{1}{2^{n-1}}$ as small as we like.

The sequence $(2 + (-1)^n \frac{1}{2^{n-1}})_{n=1}^{\infty}$

As n gets very large the positive number $\frac{1}{2^{n-1}}$ gets very small. By taking n as large as we like, we can make $\frac{1}{2^{n-1}}$ as small as we like.

Hence for very large values of n, the number $2 + (-1)^n \frac{1}{2^{n-1}}$ is very close to 2. By taking n as large as we like, we can make this number as close to 2 as we like.

We say that the sequence converges to 2, or that 2 is the limit of the sequence, and write

$$\lim_{n\to\infty} \left(2 + (-1)^n \frac{1}{2^{n-1}}\right) = 2.$$

Note: Because $(-1)^n$ is alternately positive and negative as n runs through the natural numbers, the terms of this sequence are alternately greater than and less than 2.

Convergence of a sequence : "official" definitions

Definition 73

The sequence (a_n) converges to the number L (or has limit L) if for every positive real number ε (no matter how small) there exists a natural number N with the property that the term a_n of the sequence is within ε of L for all terms a_n beyond the Nth term. In more compact language :

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ for which } |a_n - L| < \varepsilon \ \forall n > N.$$

Notes

- If a sequence has a limit we say that it converges or is convergent. If not we say that it diverges or is divergent.
- If a sequence converges to L, then no matter how small a radius around L we choose, there is a point in the sequence beyond which all terms are within that radius of L. So beyond this point, all terms of the sequence are *very close together* (and very close to L). Where that point is depends on how you interpret "very close together".

Ways for a sequence to be divergent

Being convergent is a very strong property for a sequence to have, and there are lots of different ways for a sequence to be divergent.

Example 74

1. $(\max(\{(-1)^n, 0\}))_{n=1}^{\infty}: \ 0, \ 1, \ 0, \ 1, \ 0, \ 1, \dots$
   This sequence alternates between 0 and 1 and does not approach any limit.

Ways for a sequence to be divergent

Being convergent is a very strong property for a sequence to have, and there are lots of different ways for a sequence to be divergent.

Example 74

1. $(\max(\{(-1)^n, 0\}))_{n=1}^{\infty}$: $0,\ 1,\ 0,\ 1,\ 0,\ 1, \dots$
   This sequence alternates between 0 and 1 and does not approach any limit.
2. *A sequence can be divergent by having terms that increase (or decrease) without limit.*
   $(2^n)_{n=1}^{\infty}$: $2,\ 4,\ 8,\ 16,\ 32,\ 64, \dots$

Ways for a sequence to be divergent

Being convergent is a very strong property for a sequence to have, and there are lots of different ways for a sequence to be divergent.

Example 74

1. $(\max(\{(-1)^n, 0\}))_{n=1}^{\infty}$: $0,\ 1,\ 0,\ 1,\ 0,\ 1, \dots$
   This sequence alternates between 0 and 1 and does not approach any limit.
2. *A sequence can be divergent by having terms that increase (or decrease) without limit.*
   $(2^n)_{n=1}^{\infty}$: $2,\ 4,\ 8,\ 16,\ 32,\ 64, \dots$
3. *A sequence can have haphazard terms that follow no overall pattern, such as the sequence whose nth term is the nth digit after the decimal point in the decimal representation of π.*

Convergence is a precise concept!

Remark: The notion of a convergent sequence is sometimes described informally with words like "the terms get closer and closer to L as n gets larger". It is not true however that the terms in a sequence that converges to a limit L must get progressively closer to L as n increases.

Example 75

The sequence (a_n) is defined by

$$a_n = 0 \text{ if } n \text{ is even, } a_n = \frac{1}{n} \text{ if } n \text{ is odd.}$$

This sequence begins :

$$1,\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{5},\ 0,\ \frac{1}{7},\ 0,\ \frac{1}{9},\ 0, \dots$$

It converges to 0 although it is not true that every step takes us closer to zero.

Examples of convergent sequences

Example 76

Find $\lim_{n\to\infty} \dfrac{n}{2n-1}$.

Solution: As if calculating a limit as $x \to \infty$ of an expression involving a continuous variable x, divide above and below by n.

$$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{n/n}{2n/n - 1/n} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}.$$

So the sequence $\left(\dfrac{n}{2n-1}\right)$ *converges to* $\frac{1}{2}$.

Bounded Sequences

As for subsets of $\mathbb{R}$, there is a concept of boundedness for sequences. Basically a sequence is bounded (or bounded above or bounded below) if the set of its terms, considered as a subset of $\mathbb{R}$, is bounded (or bounded above or bounded below). More precisely :

Definition 77

The sequence (a_n) is bounded above if there exists a real number M for which $a_n \leq M$ for all $n \in \mathbb{N}$.
The sequence (a_n) is bounded below if there exists a real number m for which $m \leq a_n$ for all $n \in \mathbb{N}$.
The sequence (a_n) is bounded if it is bounded both above and below.

Example 78

The sequence (n) is bounded below (for example by 0) but not above.
The sequence $(\sin n)$ is bounded below (for example by -1) and above (for example by 1).

Convergent $\Longrightarrow$ Bounded

Theorem 79

If a sequence is convergent it must be bounded.

Proof Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence with limit L.

Convergent $\Longrightarrow$ Bounded

Theorem 79

If a sequence is convergent it must be bounded.

Proof Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence with limit L.

Then (by definition of convergence) there exists a natural number N such that every term of the sequence after a_N is between $L-1$ and $L+1$.

Convergent $\implies$ Bounded

Theorem 79

If a sequence is convergent it must be bounded.

Proof Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence with limit L.

Then (by definition of convergence) there exists a natural number N such that every term of the sequence after a_N is between $L - 1$ and $L + 1$.

The set consisting of the first N terms of the sequence is a finite set : it has a maximum element M_1 and a minimum element m_1.

Convergent $\implies$ Bounded

Theorem 79

If a sequence is convergent it must be bounded.

Proof Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence with limit L.

Then (by definition of convergence) there exists a natural number N such that every term of the sequence after a_N is between $L-1$ and $L+1$.

The set consisting of the first N terms of the sequence is a finite set : it has a maximum element M_1 and a minimum element m_1.

Let $M = \max\{M_1, L+1\}$ and let $m = \min\{m_1, L-1\}$. Then (a_n) is bounded above by M and bounded below by m.

So our sequence is bounded.

Increasing and decreasing sequences

Definition 80

A sequence (a_n) is called increasing if $a_n \leq a_{n+1}$ for all $n \geq 1$.
A sequence (a_n) is called strictly increasing if $a_n < a_{n+1}$ for all $n \geq 1$.
A sequence (a_n) is called decreasing if $a_n \geq a_{n+1}$ for all $n \geq 1$.
A sequence (a_n) is called strictly decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.

Definition 81

A sequence is called monotonic if it is either increasing or decreasing.
Similar terms : monotonic increasing, monotonic decreasing, monotonically increasing/decreasing.

Note: These definitions are not *entirely* standard. Some authors use the term *increasing* for what we have called *strictly increasing* and/or use the term *nondecreasing* for what we have called *increasing*.

Examples

1. An increasing sequence is bounded below but need not be bounded above. For example

$$(n)_{n=1}^{\infty} : 1,\ 2,\ 3,\ \ldots$$

However, if a sequence is bounded and monotonic, it is convergent. This is the Monotone Convergence Theorem.

Examples

1. An increasing sequence is bounded below but need not be bounded above. For example

$$(n)_{n=1}^{\infty} : 1,\ 2,\ 3,\ \ldots$$

2. A bounded sequence need not be monotonic. For example

$$((-1)^n) : -1,\ 1,\ -1,\ 1,\ -1,\ \ldots$$

However, if a sequence is bounded and monotonic, it is convergent. This is the Monotone Convergence Theorem.

Examples

1. An increasing sequence is bounded below but need not be bounded above. For example

$$(n)_{n=1}^{\infty} : 1,\ 2,\ 3,\ \dots$$

2. A bounded sequence need not be monotonic. For example

$$((-1)^n) : -1,\ 1,\ -1,\ 1,\ -1,\ \dots$$

3. A convergent sequence need not be monotonic. For example

$$\left(\frac{(-1)^{n+1}}{n}\right)_{n=1}^{\infty} : 1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \dots$$

However, if a sequence is bounded and monotonic, it is convergent. This is the Monotone Convergence Theorem.

Examples

1. An increasing sequence is bounded below but need not be bounded above. For example

$$(n)_{n=1}^{\infty} : 1,\ 2,\ 3,\ \dots$$

2. A bounded sequence need not be monotonic. For example

$$((-1)^n) : -1,\ 1,\ -1,\ 1,\ -1,\ \dots$$

3. A convergent sequence need not be monotonic. For example

$$\left(\frac{(-1)^{n+1}}{n}\right)_{n=1}^{\infty} : 1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \dots$$

4. A montonic sequence need not be convergent, as Example 1 above shows.

However, if a sequence is bounded and monotonic, it is convergent. This is the Monotone Convergence Theorem.

The Monotone Convergence Theorem

Theorem 82

If a sequence $(a_n)_{n=1}^{\infty}$ is montonic and bounded, then it is convergent.

Proof: Suppose that (a_n) is increasing and bounded.

The Monotone Convergence Theorem

Theorem 82

If a sequence $(a_n)_{n=1}^{\infty}$ is montonic and bounded, then it is convergent.

Proof: Suppose that (a_n) is increasing and bounded.

Then the set $\{a_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$ and by the Axiom of Completeness it has a least upper bound (or supremum) L.

The Monotone Convergence Theorem

Theorem 82

If a sequence $(a_n)_{n=1}^{\infty}$ is montonic and bounded, then it is convergent.

Proof: Suppose that (a_n) is increasing and bounded.

Then the set $\{a_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$ and by the Axiom of Completeness it has a least upper bound (or supremum) L.

We will show that the sequence (a_n) converges to L.

The Monotone Convergence Theorem

Theorem 82

If a sequence $(a_n)_{n=1}^{\infty}$ is montonic and bounded, then it is convergent.

Proof: Suppose that (a_n) is increasing and bounded.

Then the set $\{a_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$ and by the Axiom of Completeness it has a least upper bound (or supremum) L.

We will show that the sequence (a_n) converges to L.

Choose a (very small) $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound for $\{a_n : n \in \mathbb{N}\}$, becasue L is the least upper bound.

The Monotone Convergence Theorem

Theorem 82

If a sequence $(a_n)_{n=1}^{\infty}$ is montonic and bounded, then it is convergent.

Proof: Suppose that (a_n) is increasing and bounded.

Then the set $\{a_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$ and by the Axiom of Completeness it has a least upper bound (or supremum) L.

We will show that the sequence (a_n) converges to L.

Choose a (very small) $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound for $\{a_n : n \in \mathbb{N}\}$, becasue L is the least upper bound.

This means there is some $N \in \mathbb{N}$ for which $L - \varepsilon < a_N$. Since L is an upper bound for $\{a_n : n \in \mathbb{N}\}$, this means

$$L - \varepsilon < a_N \leq L$$

Proof of the Monotone Convergence Theorem (continued)

$$L - \varepsilon < a_N \leq L$$

Proof of the Monotone Convergence Theorem (continued)

$$L - \varepsilon < a_N \leq L$$

Since the sequence (a_n) is increasing and its terms are bounded above by L, every term after a_N is between a_N and L, and therefore between $L - \varepsilon$ and L. These terms are all within ε of L

Using the fact that our sequence is increasing and bounded, we have

- Identified L as the least upper bound for the set of terms in our sequence
- Showed that no matter how small an ε we take, there is a point in our sequence beyond which all terms are within ε of L.

This is exactly what it means for the sequence to converge to L.

An Example (from 2011 Summer Exam)

Example 83

A sequence (a_n) *is defined by*

$$a_1 = 0, \quad a_{n+1} = \sqrt{a_n + 6} \text{ for all } n \geq 1.$$

Show that this sequence is bounded above by 3 and that it is increasing. Deduce that the sequence is convergent and find its limit.

Note: This is an example of a sequence that is defined recursively. This means that the first term is given and subsequent terms are defined (one by one) in terms of previous ones. We are not given a general formula for the nth term although one may exist.

1. 3 is an upper bound.
   Suppose that $a_k < 3$ for some k. Then

$$a_{k+1} = \sqrt{a_k + 6} < \sqrt{3 + 6} = 3.$$

   This says that if $a_k \leq 3$, then $a_{k+1} \leq 3$ also.
   Then, since $a_1 < 3$, we have $a_2 < 3$, then $a_3 < 3$, etc.

$a_1 = 0, \quad a_{n+1} = \sqrt{a_n + 6}$ for all $n \geq 1$

1. 3 is an upper bound.
   Suppose that $a_k < 3$ for some k. Then

$$a_{k+1} = \sqrt{a_k + 6} < \sqrt{3 + 6} = 3.$$

   This says that if $a_k \leq 3$, then $a_{k+1} \leq 3$ also.
   Then, since $a_1 < 3$, we have $a_2 < 3$, then $a_3 < 3$, etc.

2. The sequence is increasing
   Let $k \in \mathbb{N}$. We need to show that $a_k \leq a_{k+1}$.
   We know that $0 \leq a_k < 3$: note this implies that

$$a_k = \sqrt{a_k^2} < \sqrt{3a_k} = \sqrt{a_k + 2a_k} < \sqrt{a_k + 6} = a + k + 1.$$

   Then $a_k < a_{k+1}$ for each k, which means the sequence is increasing.

3. The sequence converges

   Since the sequence is increasing and bounded, it converges by the Monotone Convergence Theorem.

   Let L be the limit. Then, taking limits as $n \to \infty$ on both sides of the equation

$$a_{n+1} = \sqrt{a_n + 6}$$

   we find that

$$L = \sqrt{L + 6} \Longrightarrow L^2 = L + 6 \Longrightarrow (L - 3)(L + 2) = 0.$$

   Thus $L = 3$ or $L = -2$, and since all the terms of our sequence are between 0 and 3 it must be that $L = 3$.

Learning Outcomes for Section 3.2

After studying this section you should be able to

- Explain what a sequence is;
- State what it means for a sequence to be
  - convergent or divergent;
  - bounded or unbounded (above or below);
  - monotonic, increasing or decreasing.
- Give and/or identify examples of sequences with or without various properties (or combinations of properties) from the above list;
- State, prove and apply the Monotone Convergence Theorem;
- Analyze examples similar to Example 83.